GRAVITY WAVE DRAG PRODUCED BY SMALL AMPLITUDE ELLIPTICAL MOUNTAINS FOR SHEARED WIND PROFILES

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Abstract: The analytical models of Teixeira et al. (2004) and Teixeira and Miranda (2004), where the gravity wave drag exerted by a sheared stratified flow on axisymmetric or 2D mountains is calculated, are extended here to mountains with an elliptical horizontal cross section. For the simple situations considered in this study, the normalized drag depends on only two parameters: the Richardson number at the surface, *Ri*, and the horizontal aspect ratio of the mountain. For a wind that varies linearly with height, the drag always decreases as *Ri* decreases, albeit at different rates depending on the aspect ratio. For a wind that rotates with height at a constant rate maintaining its magnitude, the drag generally increases as *Ri* decreases, but if the mountain is sufficiently elongated in the surface streamwise direction, this dependence changes sign. It is also shown that flow stagnation at the surface is strongly affected by variations of the windspeed with height, but weakly affected by variations of wind direction.

***Keywords** – Gravity wave drag, Wind shear, WKB approximation*

1. INTRODUCTION

As the stably stratified tropospheric air flows over orography, internal gravity waves are generated. These waves are associated with a pressure distribution that causes a drag force of the flow on the orography. This, in turn, causes a reaction force on the atmosphere. Since the waves that generate the bulk of the drag have horizontal scales of order 10km, the drag must in general be parameterized in large-scale numerical models. For that purpose, it is useful to understand the dependence of the drag on the various physical parameters associated with the atmospheric flow and with the orography.

Existing drag parameterization schemes are based on drag expressions derived for constant atmospheric properties (windspeed and Brunt-Väisälä frequency, see Lott and Miller 1997). Nevertheless, in general the wind varies with height, and numerical studies have shown that this variation has an important impact on the drag (Grubišić and Smolarkiewicz 1997). Analytical studies addressing the problem of gravity waves have in general been limited to considering, at most, linear wind profiles, because these are the cases where an analytical solution for the waves is straightforward. However, more recently, Teixeira et al. (2004) have treated more general wind profiles by using the WKB approximation to solve the Taylor-Goldstein equation. They found that it is essential to extend the WKB solution up to second order in the small perturbation parameter in order for the wind variation with height to have any impact on the drag. Teixeira et al. (2004) assumed an axisymmetric mountain, and subsequently Teixeira and Miranda (2004) treated the case of a 2D ridge.

The present study aims to extend the model of Teixeira et al. (2004) to a mountain that is anisotropic, but not 2D. More exactly, following Phillips (1984) a mountain whose horizontal cross section is elliptical, is considered here. Drag expressions are derived for this case, and the consequences of the wind variation with height for flow stagnation at the surface are also explored using the same linear framework.

2. THEORETICAL MODEL

Steady, hydrostatic, statically stable flow over an isolated mountain is considered. If the equations of motion are linearized with respect to the perturbations induced by the mountain, combined in order to eliminate all variables except the vertical velocity perturbation, w, and if w is expressed as a Fourier integral, then it satisfies

$$\hat{w}'' + \left[\frac{N^2(k_1^2 + k_2^2)}{(Uk_1 + Vk_2)^2} - \frac{U''k_1 + V''k_2}{Uk_1 + Vk_2} \right] \hat{w} = 0, \qquad (1)$$

where $\hat{w}$ is the Fourier transform of w, N is the Brunt-Väisälä frequency, (U,V) is the unperturbed wind, (k_1,k_2) is the horizontal wavenumber of the internal gravity waves and the primes denote differentiation with respect to z. Following Teixeira et al. (2004), this equation may be solved using the WKB approximation. The appropriate solution is

$$\hat{w}(z) = \hat{w}(z=0) e^{i\int_0^z [m_0(\varepsilon\zeta) + \varepsilon m_1(\varepsilon\zeta) + \varepsilon^2 m_2(\varepsilon\zeta)] d\zeta}, \qquad (2)$$

where the vertical wavenumber $m=m_0+\varepsilon m_1+\varepsilon^2 m_2$ has been expanded up to second order in a power series of a small parameter ε. When (2) is introduced into (1), algebraic equations are obtained for m_0, m_1 and m_2. Additionally, the boundary conditions that the flow has to follow the orography at the surface,

$$\hat{w}(z=0) = iU_0 k_1 \hat{\eta} \qquad (3)$$

(where $\hat{\eta}$ is the Fourier transform of the surface elevation), and that the wave energy must radiate upward (the radiation boundary condition) must be imposed on the solution (2). This last condition determines the sign of m_0.

Since the solution is then totally specified, it remains to calculate the pressure perturbation, and the gravity wave drag, given the form of the terrain elevation. Here it will be assumed that the mountain that generates the waves has an elliptical cross section with the main axes aligned in the x and y directions. When each component of the drag (D_x, D_y) is normalized by its value in the absence of shear (D_{0x}, D_{0y}), it can be expressed as:

$$\frac{D_x}{D_{0x}} = 1 - \frac{1}{8}\left[\alpha\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2} \right) + (1-\alpha)\left(\frac{V_0'^2}{N^2} + 2\frac{V_0}{U_0}\frac{U_0'V_0'}{N^2} + 2\frac{V_0}{U_0}\frac{V_0 U_0''}{N^2} + 4\frac{V_0 V_0''}{N^2} \right) \right], \qquad (4)$$

$$\frac{D_y}{D_{0y}} = 1 - \frac{1}{8}\left[\beta\left(\frac{V_0'^2}{N^2} + 2\frac{V_0 V_0''}{N^2} \right) + (1-\beta)\left(\frac{U_0'^2}{N^2} + 2\frac{U_0}{V_0}\frac{U_0'V_0'}{N^2} + 2\frac{U_0}{V_0}\frac{U_0 V_0''}{N^2} + 4\frac{U_0 U_0''}{N^2} \right) \right], \qquad (5)$$

where (U_0,V_0) is unperturbed wind at the surface, and α and β are dimensionless coefficients dependent on the horizontal aspect ratio of the mountain $\gamma=a/b$ (a and b are, respectively, the half-widths of the mountain in the x and y directions).

Figure 1 shows the variation of the coefficients α and β with γ, for γ between 0 and 1. For symmetry reasons, $\beta(1/\gamma)=\alpha(\gamma)$, so Fig. 1(a) also represents the variation of $\beta(1/\gamma)$ and Fig. 1(b) the variation of $\alpha(1/\gamma)$ for γ between 1 and ∞. As can be seen, for $\gamma=1$ (axisymmetric mountain), $\alpha=\beta=0.75$. In this case, (4)-(5) reproduce exactly Eqs. (50)-(51) of Teixeira et al. (2004). When $\gamma=0$ (which corresponds to a 2D ridge aligned in the y direction) $\alpha=1$ and $\beta=0$. In that case (4)-(5) reduce to Eq. (13) of Teixeira and Miranda (2004).

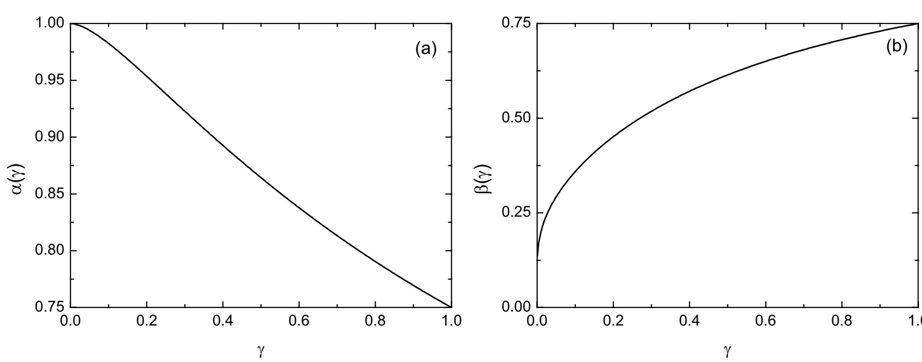

Figure 1. (a) Coefficient α as a function of the mountain horizontal aspect ratio γ,
(b) same for coefficient β.

3. RESULTS

3.1. Gravity wave drag

In order to test the drag expressions (4)-(5), two simple flows will be considered next: a backward linear wind profile, and a wind that rotates with height maintaining its magnitude. Both profiles have constant Richardson number, *Ri*, and both lead, for sufficiently high *Ri*, to total absorption of the waves at the critical levels, therefore enabling an isolation of the dynamics directly connected with the wind variation with height. The first wind profile may be expressed as

$$U = U_0 + c_1 z \,,\; V = V_0 + c_2 z \,, \tag{6}$$

where c_1 and c_2 are negative constants. The corresponding drag, which may be derived from (4)-(5) for this particular case is:

$$\frac{D_x}{D_{0x}} = 1 - \frac{\alpha}{8Ri} \,,\; \frac{D_y}{D_{0y}} = 1 - \frac{\beta}{8Ri} \,, \tag{7}$$

respectively, for a wind along x ($V_0 = 0$, $c_2 = 0$) and for a wind along y ($U_0 = 0$, $c_1 = 0$) (the other component of the drag, in each case, is zero by symmetry). Figure 2 presents the normalized drag as a function of *Ri* for $\gamma = 0.5$, in which case α=0.864 and β=0.614. Equation (7) is shown as the straight lines in Fig. 2(a). Also shown as the other curves is the exact linear drag, which results from an extension of the analytical model of Grubišić and Smolarkiewicz (1997). The symbols represent results from numerical simulations. In Fig. 2(b), similar results are shown for a wind at a 45 degree angle to both the x and the y directions ($U_0 = V_0$ and $c_1 = c_2$). In this case, the drag has two components, which are

$$\frac{D_x}{D_{0x}} = 1 - \frac{1}{16Ri}(3 - 2\alpha) \,,\; \frac{D_y}{D_{0y}} = 1 - \frac{1}{16Ri}(3 - 2\beta) \,, \tag{8}$$

respectively.

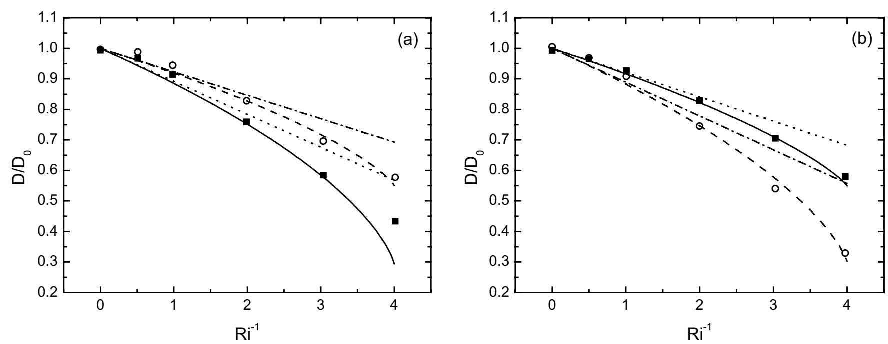

Figure 2. Normalized drag as a function of Ri^{-1} for the flow (6) and γ=0.5. Solid line: D_x/D_{0x}, from exact linear theory; dashed line: D_y/D_{0y}, from exact linear theory; dotted line: D_x/D_{0x} from WKB theory; dash-dotted line: D_y/D_{0y} from WKB theory; solid squares: D_x/D_{0x} from numerical simulations; circles: D_y/D_{0y} from numerical simulations. (a) wind along x and wind along y, (b) wind at a 45 degree angle to x and y.

The agreement of the exact linear theory with the numerical simulations (which used very nearly linear and hydrostatic conditions) is quite good. The agreement with the WKB model is a little worse, but the trends are clearly correct (the exact linear theory tends asymptotically to the WKB model for Ri>>1). While in Fig. 2(a) it can be seen that the dependence of the drag with *Ri* along the smaller axis of the mountain is stronger than along the larger axis, in Fig. 2(b) it is visible that, for an oblique wind, the dependence of the drag on *Ri* is stronger along the larger axis and weaker along the smaller axis. This, perhaps unexpected, result is connected with the opposite dependence on α and β that exists between (7) and (8), and is due to the contribution of the terms involving products of U_0' and V_0' in (4)-(5).

When a wind that rotates with height is considered, the wind profile has the form

$$U = U_0 \cos(c_3 z + c_4) \,,\; V = U_0 \sin(c_3 z + c_4) \,, \tag{9}$$

where c_3 and c_4 are constants. It is assumed here that the wind turns anti-clockwise with height (c_3>0). When the surface wind is either along x (c_4=0) or along y (c_4=π/2), from (4)-(5) the drag is given, respectively, by

$$\frac{D_x}{D_{0x}} = 1 + \frac{1}{8Ri}(3\alpha - 1)\,, \quad \frac{D_y}{D_{0y}} = 1 + \frac{1}{8Ri}(3\beta - 1)\,. \qquad (10)$$

Since α and β vary between 0 and 1, this means that it is possible for the dependence of the drag on Ri to reverse sign. However, in practice, this only happens for very elongated mountains ($\gamma = 1/36$).

3.2. Flow stagnation

Using the same linear framework as previously, and in line with the study of Smith (1989), it is possible to estimate the critical dimensionless mountain height Nh_0/U_0 (where h_0 is the dimensional mountain height) for which flow stagnation at the surface first occurs. Smith (1989) focused on flow stagnation aloft, and dismissed the importance of shear on stagnation at the surface. However, Fig. 3, which displays the critical Nh_0/U_0 for the flow (6) (Fig. 3(a)) and for the flow (9) (Fig. 3(b)) shows that the variation of the windspeed with height, and in particular the sign of this variation, has a strong impact on flow stagnation. On the other hand, the variation of the wind direction with height seems to have an almost insignificant impact.

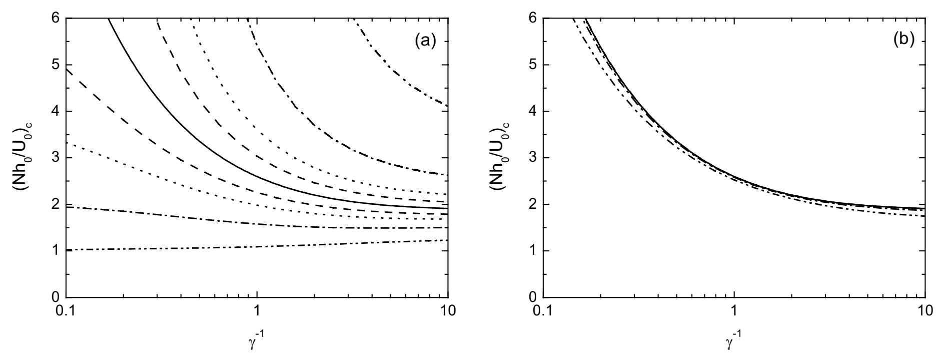

Figure 3. Critical Nh_0/U_0 for flow stagnation at the surface, as a function of the mountain aspect ratio γ for different Ri. Solid line: Ri=∞; dashed line: Ri=64; dotted line: Ri=16; dash-dotted line: Ri=4; dash-dot-dotted line: Ri=1. (a) Wind profile (6). Lower curves: forward shear, upper curves: backward shear. (b) wind profile (9).

4. CONCLUSION

The present analytical model of gravity wave drag is an extension of previously developed models to elliptical mountains, which are an important building block for representing anisotropic (realistic) orography. For the simplest flows, the normalized drag is found to depend on Ri, as in the previous studies, but also on the aspect ratio of the orography. It displays some unexpected behaviour, as when it varies more with Ri in the direction perpendicular to the mountain when the wind is along that direction than when it is parallel, but has the opposite dependence when the wind is oblique to the mountain. Or when, for a wind that rotates with height maintaining its magnitude, the drag (which generally increases as Ri decreases) changes the sign of this variation when the mountain is sufficiently elongated in the surface streamwise direction.

The present results also suggest that forward shear particularly favours flow stagnation at the surface, while backward shear inhibits this phenomenon, and wind turning with height has almost no effect on it.

Of course, all these results must be viewed with caution, since it is known that flows with critical levels are especially sensitive to nonlinear effects (even if these are small). However, the present study enables an isolation of intrinsically linear processes, which may be useful in fundamentally understanding the drag, and may help to improve its parameterization in numerical models.

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