A Theory of Gravity Wave Absorption By Boundary Layers

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Abstract: A one-layer frictional model of the boundary layer is proposed to explain the nature of gravity wave absorption seen in numerical simulations. The key result is that friction causes the phase of the BL winds to shift upstream. The associated divergence in the BL winds modulates the BL thickness in way that allows the down-going gravity wave to be absorbed. To capture the BL effect, the reflection coefficient must be considered a complex number. In a lee wave situation, the influence of the boundary layer is to decay the wave amplitude and reduce its wavelength.

Keywords: boundary layers, gravity waves

1. INTRODUCTION

The influence of internal gravity wave absorption by boundary layers (BL) was proposed by Smith et al. (2002) to explain the absence of lee waves over the Alps during the Mesoscale Alpine Programme (MAP). In the companion paper (Jiang et al. 2005), the absorption of mountain waves by a turbulent boundary layer was identified and studied with a numerical model. Using an atmospheric profile that allowed trapped lee waves to form, surface roughness and heat flux were varied systematically. With a free slip lower boundary condition, a perfect non-decaying lee wave was found. With surface friction however, the lee wave amplitude decreased exponentially downstream, suggestive of a linear absorption process. The rate of wave decay increased smoothly with the surface roughness parameter while the wavelength became significantly shorter.

The sensitivity of absorption to heat flux imposed at the lower boundary was also established by Jiang et al (2005). Surprisingly, a positive heat flux, while increasing the turbulence intensity in the BL, reduced the rate of wave decay. Conversely, a negative heat flux suppressed the turbulence level but increased the rate of wave decay. Based on this curious result, and an estimate of the direct effect of BL turbulence on wave decay, the hypothesis of simple absorption by BL turbulence was rejected.

To connect with the prior numerical experiments, we include friction parameters that mimic the effects of surface roughness and heat flux. The formulation retains the two essential elements of the boundary layer that sets it apart from the free atmosphere: slower wind and frictional dissipation.

2. MODEL FORMULATION

We begin by imagining a thin layer of fluid near the ground with thickness $H+\eta$ and speed $U_B = \overline{U}_B + u'_B$ beneath a deep free atmosphere with wind speed $U = \overline{U} + u'$. The pressure (P) imposed by the free atmosphere penetrates the thin boundary layer. The coefficient C_B represents friction at the surface and C_T represents friction between the BL and the free atmosphere. Increased surface roughness should increase C_B. According to Monin-Obukhov scaling, upper BL turbulence will respond to surface heat flux. Positive heat flux will cause thermal convection that will increase C_T. Negative heat flux will stabilize the upper BL and reduce C_T. The steady momentum equation for the BL is

$$U_B \frac{dU_B}{dx} = -\frac{dP}{dx} - C_B U_B + C_T (U - U_B) \qquad (1)$$

while the wind in the free atmosphere obeys

$$U\frac{dU}{dx}=-\frac{dP}{dx}\quad , \tag{2}$$

neglecting the mean-state pressure gradient and Coriolis forces. In the absence of any disturbance, $P, U=\overline{U}, U_B=\overline{U}_B$ are constant so from (1)

$$\overline{U}_B=\frac{C_T}{C_B+C_T}\overline{U}=\frac{r_C}{1+r_C}\overline{U} \tag{3}$$

where the friction ratio $r_C=C_T/C_B$. As r_C increases, the BL wind speed $\overline{U}_B\to\overline{U}$. Assuming that the perturbations are periodic of the form $f(x)=\mathrm{Re}[\hat{f}e^{ikx}]$, (1, 2) give

$$ik\overline{U}_B\hat{u}_B=-ik\hat{P}-C_B\hat{u}_B+C_T(\hat{u}-\hat{u}_B) \tag{4}$$

and

$$ik\overline{U}\hat{u}=-ik\hat{P}\ . \tag{5}$$

According to (4, 5) the BL wind speed can respond more strongly to an imposed pressure gradient than the free atmosphere because of its smaller advective velocity (i.e. $U_B<U$); but it is also limited by the friction terms. The wind response ratio ($\hat{R}$) can be found from (4, 5)

$$\hat{R}=\frac{\hat{u}_B}{\hat{u}}=\frac{ik\overline{U}+C_T}{ik\overline{U}_B+C_B+C_T} \tag{6}$$

According to (6), the wind speed in the BL responds to free atmosphere wind speed perturbations with a different amplitude and phase. Two non-dimensional parameters determine the response:

$$r_k=|k|\overline{U}/C_B \text{ and } r_C=C_T/C_B\,.$$

The former parameter resembles a "Reynolds Number" as it is the ratio of inertia to friction. When r_C is large, (3) and (6) demand that $\overline{U}_B=\overline{U}$ and $\hat{R}=1$ so that there is no BL effect. When $r_C\sim 1$, either a large r_k or small r_k makes $\hat{R}$ real so that there is no phase difference between the free atmosphere forcing and BL response. When r_k is near unity, $\hat{R}$ is complex. Note that when $\overline{U}, \overline{U}_B, \alpha, \gamma$ are all positive, complex $\hat{R}$ lies in the 1st or 4th quadrant depending on whether k is positive or negative. In both cases, the phase shift represents an upstream shift to the BL wind perturbation. Physically, the phase shift is associated with a three-way balance in (4); acceleration, pressure gradient and friction.

3. WAVE REFLECTION

The nature of wave reflection can be related to the wind response (6) by using mass conservation in the lower layer

$$(H+\eta)(\overline{U}_B+u'_B)=C \tag{7}$$

where C is some constant, so that the Fourier amplitudes obey $\hat{\eta}=-H\frac{\hat{u}_B}{\overline{U}_B}$. As the upper fluid flows over the BL, its vertical velocity is $\hat{w}=ik\overline{U}\hat{\eta}$. Using (7), we can derive expressions for the momentum and energy flux at the BL top. From (5,6,7) the complex "compliance" coefficient $\hat{\beta}$, defined by

$$\hat{\beta}=\frac{\hat{w}}{\hat{P}}=\frac{ikH}{\overline{U}_B}\hat{R} \tag{8}$$

describes the vertical motion induced at the BL top by imposed pressure disturbances. In (8), the sign of $\mathrm{Re}[\hat{\beta}]$ controls the sign of the energy flux. The energy flux (EF) across the BL top arising from each Fourier mode is

$$EF=\frac{\rho_0}{2}\mathrm{Re}[\hat{P}\hat{w}^*]=\frac{\rho_0}{2}|\hat{P}|^2\,\mathrm{Re}[\hat{\beta}]\quad , \tag{9}$$

where we have used (8), and several rules of complex multiplication. The asterisk indicates the complex conjugate. When $|\hat{\beta}|=0$, (8) becomes the usual rigid boundary condition ($\hat{w}=0$) without energy loss at the surface. When $\mathrm{Re}[\hat{\beta}] < 0$, the energy flux across the boundary layer top is negative implying that wave energy is removed by the boundary layer. According to (8), this requires that the wind response ratio $\hat{R}$ lies in the first quadrant for k>0. The BL wind disturbance must lead the free atmosphere. Condition (8), with $\mathrm{Re}[\hat{\beta}] > 0$ applied at the top of the domain would be equivalent to a radiation condition aloft (Klemp and Durran, 1983).

As an example, consider a free atmosphere with mean wind speed of $\overline{U} = 10ms^{-1}$ over a boundary layer 100 meters thick with equal friction coefficients $C_B = C_T = 0.001s^{-1}$. The free atmosphere carries a disturbance with a wavelength of $\lambda = 12km$ so $k = 2\pi/\lambda = 0.5 \bullet 10^{-3} m^{-1}$. From (3), the mean BL speed is $5ms^{-1}$. From (6), the wind response ratio is

$$\hat{R} = \frac{5i+1}{2.5i+2} = 1.41 + 0.73i = 1.59e^{i\theta} \tag{10}$$

where $\theta = 27.3^0$. According to (10), the amplitude of wind speed oscillations in the BL is 59% larger than that in the free atmosphere, and it is phase shifted upstream by 27.3 degrees

In stratified flow with standing waves, the up and down going waves can be identified in the expression

$$\hat{w}(z) = \hat{A}e^{imz} + \hat{B}e^{-imz} \tag{11}$$

where $m = [l^2 - k^2]^{1/2}\,\mathrm{sgn}(k)$ and $l^2 = N^2/\overline{U}^2$. In this case, the wave reflection coefficient is the negative of the ratio between the up and down-going wave coefficients: i.e $\hat{q} = -\hat{A}/\hat{B}$. Using (5, 8) along with the continuity equation ($ik\hat{u} + \hat{w}_z = 0$), gives $\hat{\beta}\overline{U}\hat{w}_z = ik\hat{w}$ and thus with (12)

$$\hat{q} = \frac{1+\gamma}{1-\gamma} \tag{12}$$

where $\gamma = \hat{\beta}\overline{U}m/k = i(\overline{U}/\overline{U}_B)Hm\hat{R}$ (Jiang et al., 2005). In (12), the gravity wave vertical wavenumber (m) enters because it influences the magnitude of the free atmosphere pressure oscillation. As γ always lies in the 2nd quadrant, (13) gives $|q|<1$. Thus, in spite of the ambient shear in our formulation, no over-reflection is found (Lindzen and Tung, 1978). Choosing $N=0.01s^{-1}$ and k>0, our example has $m=0.001m^{-1}$ and $\gamma = 0.2i\hat{R}$. From (10 and 12)

$$\hat{q} = 0.644 + 0.406i = 0.761e^{i\theta} \tag{13}$$

with $\theta = 32^0$. Thus the amplitude of the reflected wave is 76% of the incident amplitude. The phase of the reflected wave is shifted by 32 degrees. For negative k of the same magnitude, $\hat{q}$ is the complex conjugate of (13).

4. TRAPPED LEE WAVE DECAY

The influence of a wave reflection coefficient ($\hat{q}$) on a trapped lee wave in a rigid waveguide of depth D can be determined using (11) and a rigid lid condition w=0 at z=D; giving the lee wave eigenvalue condition

$$\hat{q} = e^{-2im(k)D}. \tag{14}$$

For each value of $\hat{q}$ at the lower boundary, (14) can be solved for complex k. With perfect reflection ($\hat{q} = 1$) we define $k = k_0$, $m_0 D = n\pi$ ($n = 1,2..$). An absorbing BL induces a change in the wavenumber $\Delta k = k - k_0$ due to the change in reflection coefficient $\Delta\hat{q} = \hat{q} - 1$. Taking the derivative ($d\hat{q}/dk$) of (14), we obtain

$$\Delta k = \frac{m_0 \Delta\hat{q}}{2ik_0 D} \ . \tag{15}$$

$\Delta\hat{q}$ lies in the second quadrant (for $k_0 > 0$), so (15) puts Δk in the first quadrant (i.e. with positive real and imaginary parts). Due the BL effect then, a lee wave described locally by

$$w(x,z) = \mathrm{Re}[\hat{w}(z)e^{i(k_0+\Delta k)x}] \tag{16}$$

has a new wavenumber $k_0 + \Delta k$ with a positive imaginary part implying decay and an increased magnitude of its real part, implying a shortened wavelength. The shortened wavelength arises from phase advancement during each reflection event. The decay and wavelength change are illustrated in Fig. 1, using a full FFT solution. Jiang et al. (2005) found similar results in deep two-layer atmospheres.

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REFERENCES

Jiang, Q., J.D. Doyle, R.B. Smith, 2005, Interaction between trapped waves and boundary layers, (Submitted to *J. Atmos. Sci.*)

Klemp J.B. and D. R. Durran. 1983: An Upper Boundary Condition Permitting Internal Gravity Wave Radiation in Numerical Mesoscale Models. *Mon. Wea. Rev.*, **111**, 430–444.

Lindzen, R.S. and K.K. Tung. 1978: Wave Over-reflection and Shear Instability. *Journal of the Atmospheric Sciences*: Vol. 35, No. 9, pp. 1626–1632.

Nappo, C J., G. Chimonas, 1992, Wave Exchange between the Ground Surface and a Boundary-Layer Critical Level, *J. Atmos. Sci*, **49**: 1075-1091

Smith, R.B., S. Skubis, J.D. Doyle, A.S. Broad, C. Kiemle, H. Volkert, 2002, Mountain Waves over Mont Blanc: Influence of a stagnant boundary layer, *J. Atmos. Sci.*, **59**, 2073-2092.

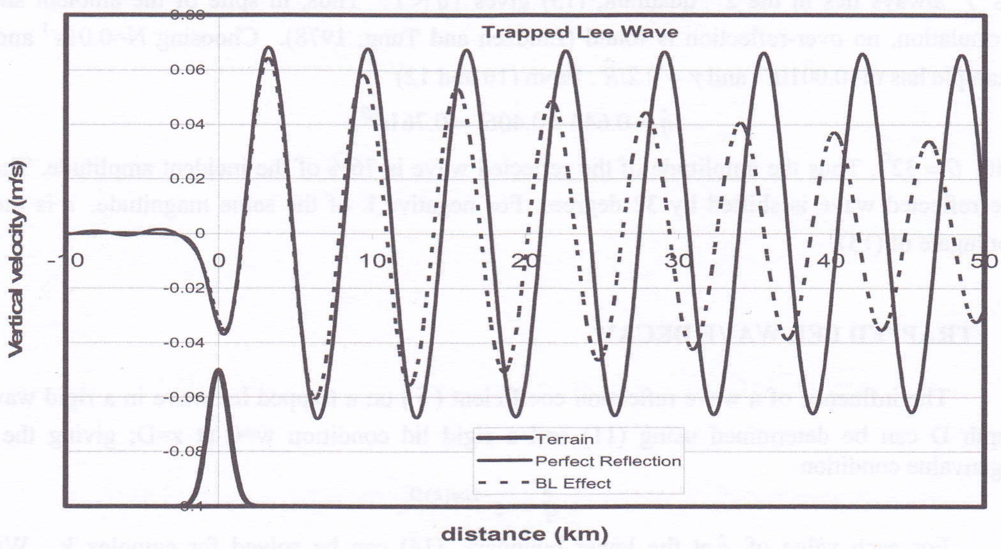

Figure 1. Influence of BL absorption at the lower boundary on a trapped lee wave. An FFT solution to waves in a rigid wave guide with depth of D=3000m, U=7m/s, N=0.01s^{-1}, $\Delta x = 200m$. Over 50km downwind of the ridge, about eight wave cycles are seen. Two reflection coefficients are shown: $\hat{q} = .995 + 0i$ and $\hat{q} = .88 + .25i$. The change in the lee wave agrees with the asymptotic result (16). The BL effect decays the wave and shortens its wavelength.