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LIFTING A CIRCULAR MEMBRANE BY UNITARY FORCES

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ABSTRACT. Let Ω be a convex membrane. We lift certain parts Γ of its boundary by means of unitary forces while the remaining parts are maintained at level 0. Call $u[\Gamma]$ the position that the such lifted membrane assumes. When the parts Γ are varying on $\partial\Omega$ so that their total length C is preserved, it has been conjectured that the functional $\Gamma \mapsto \|u(\Gamma)\|_p$ attains its maximum value for a certain connected arc of length C. In this paper we present a proof of this conjecture for the case in which Ω is a circle and $p = 1$.

1. INTRODUCTION

Let us consider a convex membrane represented by an open and convex set $\Omega \subseteq \mathbb{R}^2$, which is maintained at level 0 on certain parts Γ_0 of its boundary, while it is lifted by unitary normal forces at the remaining portions $\Gamma_1 = \partial B_1(0)\backslash\Gamma_0$. As it is well known, the displacement u of the membrane is the solution to the following mixed boundary problem for the Laplacian:

$$(1) \qquad \left| \begin{array}{ll} \Delta u(x) = 0, & x \in \Omega \\ u(x) = 0, & x \in \Gamma_0 \\ \frac{\partial u(x)}{\partial r} = 1 & x \in \Gamma_1 \end{array} \right. .$$

If Ω has a sufficiently smooth boundary, it is known that problem (1) admits a classical solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ (see [11] and the references therein). Henceforth we write $u = u[\Gamma_1]$ to emphasize the dependence of the solution to problem (1) on the choice of Γ_1. For a given number $0 < C < |\partial\Omega|$, we define the family $\mathcal{F}(\Omega, C)$ as composed by those Γ_1 which are finite subsets of $\partial\Omega$ with total length equals to C. In [3] was conjectured that, for two fixed numbers C and p such that $0 < C < |\partial\Omega|$, $1 \leq p \leq +\infty$, the maximum of the $L^p(\partial\Omega)$ norm of $u[\Gamma_1]$,

$$(2) \qquad \|u[\Gamma_1]\|_{L^p(\partial\Omega)} = \begin{cases} \left(\int_{\partial\Omega} |u[\Gamma_1]|^p \, ds\right)^{1/p}, & 1 \leq p < +\infty \\ \sup_{\partial\Omega} |u[\Gamma_1]|, & p = +\infty \end{cases} ,$$

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when Γ_1 is varying on $\mathcal{F}(\Omega, C)$, is realized by a *connected* arc Γ_1^* of length C. In particular, for $p = +\infty$, the conjecture asserts that the maximum height reached by a convex membrane which is lifted by unitary forces on some portions Γ_1 of its boundary with constant total measure C, occurs when the membrane is lifted on a certain connected arc $\Gamma_1^* \subseteq \partial\Omega$ of length C.

In this paper we show that the conjecture is true for a circular membrane $\Omega = B_1(0)$ and $p = 1$. Concretely, denoting by $u[\Gamma_1]$ the solution to problem (1) for $\Omega = B_1(0) = B_1$, we prove that the following result.

Theorem 1. *For any number $0 < C < 2\pi$, we have*

$$\max_{\Gamma_1 \in \mathcal{F}(B_1(0), C)} u[\Gamma_1](0) = u[\Gamma_1^*](0), \tag{3}$$

where Γ_1^* is an arc of length C.

We emphasize that the maximum in (3) is taken over the family $\mathcal{F}(B_1(0), C)$ of finite subset of arcs of $\partial B_1(0)$ with total measure equals to C. Since $u[\Gamma_1]$ is harmonic in $B_1(0)$, the Mean Value Theorem provides

$$\int_{\partial B_r(0)} u[\Gamma_1]\, ds = 2\pi r u[\Gamma_1](0), \tag{4}$$

for every $0 < r < 1$. Thus, realizing that $u[\Gamma_1]$ is continuous up to the boundary, we can take limits for $r \uparrow 1$ in (3) to obtain

$$\int_{\partial B_1} u[\Gamma_1]\, ds = 2\pi u[\Gamma_1](0); \tag{5}$$

but, as can be easily derived from the Hopf's lemma ([4], pg. 34), the solution $u[\Gamma_1]$ is non-negative and then

$$\|u[\Gamma_1]\|_{L^1(\partial B_1)} = \int_{\partial B_1} u[\Gamma_1]\, ds;$$

therefore, $\|u[\Gamma_1]\|_{L^1(\partial B_1)} = 2\pi u[\Gamma_1](0)$ and it is concluded that Theorem 1 proves the conjecture for $\Omega = B_1$ and $p = 1$, as we said above.

Other statements equivalent to Theorem 1 can be easily obtained. For example, an argument like the previous one, but using the "volumetric" version of the Mean Value Theorem instead of (4), shows that the height at the origin $u[\Gamma_1](0)$ of the solution $u[\Gamma_1]$ can be replaced by the $L^1(B_1)$ norm $\|u[\Gamma_1]\|_{L^1(B_1)}$. This means that the maximum of the mean height of a circular membrane lifted by unitary forces on portions of constant length C of its boundary, is attained when an arc of length C is lifted. At the light of this interpretation, that also the potential energy of a membrane is maximum when it is lifted at a connected arc should not be a surprise. In fact, by the

Green's formulas, for $0 < r < 1$ we have

$$(6) \qquad 0 = \int_{B_r} u[\Gamma_1]\, \Delta u[\Gamma_1]\, dx = \int_{\partial B_r} u[\Gamma_1] \frac{\partial u[\Gamma_1]}{\partial r}\, ds - \int_{B_r} |\nabla u[\Gamma_1]|^2\, dx.$$

If we take limits for $r \uparrow 1$ in (6), using the conditions satisfied at the boundary by $u[\Gamma_1]$ we find

$$\int_{\partial B_1} u[\Gamma_1]\, ds = \int_{B_1} |\nabla u[\Gamma_1]|^2\, dx,$$

which, after Theorem 1 and the previous remarks, says that also the Dirichlet integral of $u[\Gamma_1]$; i.e. the potential energy of the membrane, is maximized by a connected arc Γ_1^* of length C when Γ_1 is varying on $\mathcal{F}(B_1(0), C)$. Now, we collect all these equivalent statements of Theorem 1 in the following corollary.

Corollary 2. *When Γ_1 varies on the family $\mathcal{F}(B_1(0), C)$, the functionals*

i): $\Gamma_1 \mapsto \|u[\Gamma_1]\|_{L^1(\partial B_1)}$,
ii): $\Gamma_1 \mapsto \|u[\Gamma_1]\|_{L^1(B_1)}$,
iii): $\Gamma_1 \mapsto \int_{B_1} |\nabla u[\Gamma_1]|^2\, dx$,

attain their respective maximum when Γ_1 is an arc of length C.

In the next section, a less immediate equivalence connected to the capacity of finite unions of closed arcs of the unit circumference serves to the purpose of proving Theorem 1. Some unsolved problems related to the aforementioned general conjecture are presented in the final section.

2. Proof of Theorem 1

The proof we shall give for Theorem 1 rely on the following result.

Theorem 1. *A closed set Γ on S^1, the unit circumference, such that $|\Gamma| = C$ has a capacity at least equal to $\sin(C/4)$, the capacity corresponding to an arc of length C.*

L. V. Ahlfors has attributed Theorem 1 to A. Beurling in [2], pgs. 30-36, where a proof is provided by employing a symmetrization argument. The clue of this argument is the fact that Dirichlet integral $D(u) = \int_{B_1} |\nabla u|^2\, dx$ does not increases by circular symmetrization ([2], see also [5], pg. 94). A different proof using tools from the Geometric Theory of Functions can be found in Chap. 11 of [9] (see also Problem 36, pg. 146, of [7]).

By using Corollary 2-**iii)**, a direct proof of Theorem 1 using symmetrization techniques is feasible. Nevertheless, we take another way consisting in to exhibit the equivalence of theorems 1 and 1. To this end, we need to expose first some concepts and results on capacities and Green's functions.

Let us begin by reminding the concept of capacity of a compact set $\Gamma \subseteq \mathbb{R}^n$. If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain such that $\partial\Omega$ is sufficiently regular, there exists a harmonic function u defined on the complement $\mathbb{R}^n \backslash \Omega$ which verifies $u|_{\partial\Omega} = 1$ and $\lim_{|x|\to\infty} u(x) = 0$ (see [4], pg. 27; [6], pg. 330). The *capacity* of Ω, denoted by $cap(\Omega)$, is then defined as follows:

$$cap(\Omega) = -\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = \int_{\mathbb{R}^n\backslash\Omega} |\nabla u|^2\, dx, \tag{7}$$

where n is the unit outward normal to $\partial\Omega$. For any compact set E, the capacity $cap(E)$ is defined as the limit $\lim_{n\uparrow+\infty} cap(\Omega_n)$, where $\{\Omega_n\}$ is a sequence of nested domains with smooth boundary such that $\cap_{n=1}^{\infty}\Omega_n = E$. The notion of capacity expressed by (7) corresponds to the idea of capacity of the isolated conductor Ω as classically arose in Electrostatic; i.e., to the ratio of the electrical charge in equilibrium on Ω (given by $-\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds$) to the value of the potential at its surface ($u = 1$ on $\partial\Omega$). However, many other equivalent definitions have been given for the capacity of a compact set (see [2], [4], [7], [9], [10] and [12]).

Particularly relevant for our developments are the relationships between capacity and Green's functions. A readable presentation of this topic can be found in the books [2], [9] and [10]. Here, we limit ourselves to point out that if $E \subseteq \mathbb{C}$ is compact and Ω is its *outer domain*; that is, the component of $\overline{\mathbb{C}}\backslash E$ that contains ∞ and if we assume that Ω is regular; namely, if Ω is connected and bounded by a finite number of piecewise analytic Jordan curves; then Ω admits a Green's function g_E with pole at ∞ ([1], [8]); i.e., a function g_E which is harmonic in Ω, it satisfies $g_E|_{\partial E} = 0$ and its asymptotic behaviour at ∞ is of the form

$$g_E(z) = \ln|z| + \gamma + O(|z|^{-1}) as z \to \infty. \tag{8}$$

Indeed, the Green's function g_E exists under much less restrictive conditions on the domain Ω (see [10], Theorems 9.7 and 9.8, pgs. 205-207). The constant γ that appears at the right hand side of (8), known as *Robin constant* of E, is related to the capacity of E through ([2], [9], [10])

$$\gamma = -\ln cap(E). \tag{9}$$

Now we turn to consider the unit circle $B_1(0)$ with its boundary split in two finite families of arcs Γ_0 and $\Gamma_1 = \partial B_1(0)\backslash\Gamma_0$. A simple relationship between $u[\Gamma_1]$ and g_{Γ_0} is established by our next result.

Lemma 2. *Let $u[\Gamma_1]$ and g_{Γ_0} respectively be the solution to problem (1) for $B_1(0)$ and the Green's function of Γ_0, then*

$$u[\Gamma_1] = 2g_{\Gamma_0}|_{B_1(0)}. \tag{10}$$

Furthermore,

$$u[\Gamma_1](0) = 2g_{\Gamma_0}|_{B_1(0)}(0) = -2\ln cap(\Gamma_0). \tag{11}$$

Proof. We know that $2g_{\Gamma_0}|_{B_1(0)}$ is harmonic in $B_1(0)$ and that it vanishes on Γ_0. Then, to prove the equality (10) it will be sufficient to show that $\partial(2g_{\Gamma_0})/\partial r = 1$ on Γ_1. With this purpose, we define the function $h_0(z) = g_{\Gamma_0}(z) - g_{\Gamma_0}(1/\overline{z})$ which is harmonic in $\mathbb{C}\backslash(\Gamma_0 \cup \{0\})$ and, in view of (8), it has logarithmic singularities at the points $z = 0$ and $z = \infty$. Moreover, we have $h_0(e^{i\theta}) = g_{\Gamma_0}(e^{i\theta}) - g_{\Gamma_0}(e^{i\theta}) = 0,\ 0 \leq \theta < 2\pi$. Hence, $h_1(z) = h_0(z) - \ln|z|$ is harmonic in $\overline{\mathbb{C}}\backslash\Gamma_0$ and it vanishes on Γ_0 and also for $z \to \infty$; thus, the maximum principle shows that $h_1 = 0$, that is

$$g_{\Gamma_0}(z) - g_{\Gamma_0}(1/\overline{z}) = \ln|z|\,,\ z \in \overline{\mathbb{C}}\backslash\Gamma_0. \tag{12}$$

By setting $z = re^{i\theta}$ in (12), we obtain

$$g_{\Gamma_0}(re^{i\theta}) - g_{\Gamma_0}(r^{-1}e^{i\theta}) = \ln r,\ re^{i\theta} \in \overline{\mathbb{C}}\backslash\Gamma_0$$

and differentiating with respect to r,

$$\frac{\partial g_{\Gamma_0}(re^{i\theta})}{\partial r} + \frac{1}{r^2}\frac{\partial g_{\Gamma_0}(r^{-1}e^{i\theta})}{\partial r} = \frac{1}{r},\ re^{i\theta} \in \overline{\mathbb{C}}\backslash\Gamma_0.$$

If we take $re^{i\theta} \in \Gamma_1$ in the last identity, we finally arrive to

$$2\frac{\partial g_{\Gamma_0}(e^{i\theta})}{\partial r} = 1,$$

which shows that $2g_{\Gamma_0}|_{B_1(0)}$ is the solution $u[\Gamma_1]$ to problem (1).

In order to prove (11), from (8) and (9) we deduce that

$$g_{\Gamma_0}(z) = \ln|z| - \ln cap(\Gamma_0) + O(|z|^{-1}) as z \to \infty;$$

therefore,

$$\begin{aligned} 0 &= \lim_{z\to 0}\left(g_{\Gamma_0}(z) - g_{\Gamma_0}(1/\overline{z}) - \ln|z|\right) \\ &= g_{\Gamma_0}(0) + \lim_{z\to 0}\left(-\ln|1/\overline{z}| + \ln cap(\Gamma_0) + O(|z|) - \ln|z|\right) \\ &= g_{\Gamma_0}(0) + \ln cap(\Gamma_0); \end{aligned}$$

i.e., $g_{\Gamma_0}|_{B_1(0)}(0) = -\ln cap(\Gamma_0)$. Together with (10), this equality completes the proof. □

Now we are in situation to prove Theorem 1.

Proof of Theorem 1. On one hand, (11) shows that

$$u[\Gamma_1](0) = -2\ln cap(\Gamma_0)$$

and, on the other, Theorem 1 ensures that $cap(\Gamma_0) \geq \sin(|\Gamma_0|/4) = cap(\Gamma_0^*)$, where Γ_0^* is an arc of length $|\Gamma_0| = 2\pi - C$. We conclude that

$$u[\Gamma_1](0) \leq -2\ln cap(\Gamma_0^*) = -2\ln\sin\left(\frac{2\pi - C}{4}\right) = -2\ln\cos\left(\frac{C}{4}\right) = u[\Gamma_1^*](0),$$

where Γ_1^* is an arc of length C. □

3. Open problems

The conjecture relative to the functional $\mathcal{F}(B_1(0), C) \ni \Gamma_1 \mapsto \|u[\Gamma_1]\|_p$ we state in the introduction remains open for $p \neq 1$. The general case of the conjecture for a convex domain Ω and every $1 \leq p \leq +\infty$ is also an open problem . The following generalization of the Beurling's result, Theorem 1, seems to be supportable: let $\gamma \subseteq \mathbb{R}^2$ be a closed convex curve and let $0 < C < |\gamma|$; there exists an arc Γ^* of γ such that if Γ denotes a finite number of arcs of γ with total length C, then $cap(\Gamma) \geq cap(\Gamma^*)$.

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