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A SPECIAL CLASS OF UNIVALENT OPERATORS

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ABSTRACT. Let $H(U)$ be the space of analytic functions in the unit disk U and let $h \in H(U)$. We define a subset $\mathcal{K}_{(\beta,\gamma),h} \subset H(U)$ such that the operator $A_{(\beta,\gamma),h} : \mathcal{K}_{(\beta,\gamma),h} \longrightarrow H(U)$ given by

$$A_{(\beta,\gamma),h}(f)(z) = \left[\frac{\beta+\gamma}{h^{\gamma}(z)} \int_0^z f^{\beta}(t)h^{\gamma-1}(t)h'(t)dt\right]^{1/\beta}$$

is well defined. Then we determine a class of functions whose images by $A_{(\beta,\gamma),h}$ operators are univalent. In addition, we give some particular cases of our main result obtained for appropriate choices of h, β and γ.

1. INTRODUCTION

Let $H(U)$ be the space of all analytic functions in the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$ and let $h \in \mathcal{A} = \{h \in H(U) : h(0) = 0, h'(0) \neq 0, h(z)h'(z) \neq 0, for\ 0 < |z| < 1\}$. For $f \in \mathcal{K}_{(\beta,\gamma),h} \subset H(U)$ let $F = A_{(\beta,\gamma),h}(f)$ where

$$F(z) = \left[\frac{\beta+\gamma}{h^{\gamma}(z)} \int_0^z f^{\beta}(t)h^{\gamma-1}(t)h'(t)dt\right]^{1/\beta}, \ \beta,\gamma \in \mathbf{C}. \tag{1.1}$$

This type of integral operators and different particular cases were studied in several papers like [1], [2], [3], [6], [7], [8] and others.

In the present paper , first we will determine sufficient conditions on h and the correspondent classes $\mathcal{K}_{(\beta,\gamma),h}$ such that the operator given by (1.1) will be well defined. Then we will find a class of functions whose images by $A_{(\beta,\gamma),h}$ operator are univalent in U and in addition some particular cases obtained for different choices of h, β and γ will be given.

2. PRELIMINARIES

In order to prove our main results, we will need the following definitions and lemmas presented in this section.

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Like in [8], let $c \in \mathbf{C}$ with $Re\ c > 0$ and let $N = N(c) = \dfrac{|c|\sqrt{1 + 2\,Re\ c} + Im\,c}{Re\ c}$. Considering the univalent function $k(z) = \dfrac{2Nz}{1 - z^2}$, we define the *"open door"* function

$$R_c(z) = k\left(\frac{z + b}{1 + \bar{b}z}\right),\ z \in U. \tag{2.1}$$

Note that R_c is univalent in U, $R_c(0) = c$ and $R_c(U) = k(U)$ is the complex plane slit along the half lines $Re\ w = 0, Im\ w \geq N$ and $Re\ w = 0, Im\ w \leq -N$.

For $f,\ g \in H(U)$ we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subset g(U)$.

We denote by $A = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$ and let $D = \{\phi \in H(U) : \phi(z) \neq 0\ for\ z \in U, \phi(0) = 1\}$.

Lemma 1. *[4] Let $\phi, \Phi \in D$ and let $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ with $\beta \neq 0, \alpha + \delta = \beta + \gamma$ and $Re\ (\alpha + \delta) > 0$. If $f \in A$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta}(z),$$

where R_c is defined by (2.1) and if the function F is defined by

$$F = A_{\beta,\gamma}(f)\ where\ A_{\beta,\gamma}(f)(z) = \left[\frac{\beta + \gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{1/\beta} \tag{2.2}$$

then

$$F \in A,\ \frac{F(z)}{z} \neq 0,\ z \in U\ and\ Re\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

(All powers in (2.2) are principal ones.)

A function $f \in A$ is called *a starlike function of order* α, $\alpha < 1$, if $Re\ \dfrac{zf'(z)}{f(z)} > \alpha$ for all $z \in U$ and we denote by $S^*(\alpha)$ the class of all these functions.

Lemma 2. *[9] Let $\beta > 0, \beta + \gamma > 0$ and consider the integral operator $A_{\beta,\gamma}$ defined by (2.2). If $\alpha \in \left[-\dfrac{\gamma}{\beta}, 1\right)$, then the order of starlikeness of the class $A_{\beta,\gamma}(S^*(\alpha))$, i.e. the largest number $\delta = \delta(\alpha;\beta,\gamma)$ such that $A_{\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta)$ is given by $\delta(\alpha;\beta,\gamma) = inf\{Re\ q(z) : z \in U\}$, where*

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \qquad and \qquad Q(z) = \int_0^1 \left(\frac{1 - z}{1 - tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1}dt.$$

Moreover if $\alpha \in [\alpha_0, 1)$, *where* $\alpha_0 = max\left\{\frac{\beta-\gamma-1}{2\beta}; -\frac{\gamma}{\beta}\right\}$ *and* $g = A_{\beta,\gamma}(f)$ *where* $f \in S^*(\alpha)$, *then*

$$Re\,\frac{zg'(z)}{g(z)} > \delta(\alpha;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1, 2\beta(1-\alpha), \beta+\gamma+1; \frac{1}{2})} - \gamma\right], \; z \in U,$$

where ${}_2F_1$ *represents the hypergeometric function.*

The next lemma concerns subordination (or Loewner) chains. A function $L(z;t)$, $z \in U$, $t \geq 0$ is called ***a subordination chain*** if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,+\infty)$ for all $z \in U$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$ [10, p. 157].

Lemma 3. *[10, p. 159] The function* $L(z;t) = a_1(t)z + \cdots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t\to+\infty} |a_1(t)| = +\infty$ *is a subordination chain if and only if*

$$Re\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \; z \in U, \, t \geq 0.$$

The next lemma is a slight modification of a result of K. Sakaguchi which provides a sufficient condition for univalence.

Lemma 4. *[11, Corollary 3] Let* $Re\,\beta > -\frac{1}{2}$ *and for* $F \in H(U)$, *with* $F'(0) \neq 0$ *let*

$$J(\beta,F)(z) = (\beta-1)\frac{zF'(z)}{F(z)} + \frac{zF''(z)}{F'(z)} + 1. \tag{2.3}$$

If $Re\,J(1,F)(z) > -\frac{1}{2}$, $z \in U$ *or* $Re\,J(\beta,F)(z) > -\frac{1}{2}$, $z \in U$ *when* $\beta \neq 1$ *and* $F(0) = 0$, *then* F *is univalent in* U.

The last lemma deals with the univalent solutions of *Briot-Bouquet differential equations* and represents a simplified form of Theorem 1 from [5].

Lemma 5. *[5,Theorem 1] Let* $\beta, \gamma \in \mathbf{C}$ *with* $\beta \neq 0$ *and let* $h(z) = c + h_1 z + \cdots$ *be analytic in* U. *If* $Re\,[\beta h(z) + \gamma] > 0$, $z \in U$ *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \; \text{with } q(0) = c,$$

is analytic in U *and the solution satisfies* $Re\,[\beta q(z) + \gamma] > 0$, $z \in U$.

3. Main Results

Our first result gives us sufficient conditions on h function such that the integral operator (1.1) is well defined.

Theorem 1. *Let $\beta,\gamma\in\mathbf{C}$ with Re $(\beta+\gamma)>0$ and let $h\in\mathcal{A}$. Then the integral operator given by (1.1) is well defined on the subset*

$$\mathcal{K}_{(\beta,\gamma),h}=\left\{f\in H(U):f(0)=0,f'(0)\neq 0,\beta\frac{zf'(z)}{f(z)}+J(\gamma,h)(z)\prec R_{\beta+\gamma}(z)\right\}.$$

Proof. In order to prove our theorem we will use Lemma 1 for $\alpha:=\beta$ and $\gamma:=\delta$. Taking in this lemma $\phi(z)=\left(\frac{h(z)}{z}\right)^{\gamma-1}h'(z)$ and $\Phi(z)=\left(\frac{h(z)}{z}\right)^{\gamma}$, since $h\in\mathcal{A}$ we easily deduce $\phi,\Phi\in D$.

A simple computation shows that $\frac{z\phi'(z)}{\phi(z)}=J(\gamma,h)(z)-\gamma$ hence the condition

$$\beta\frac{zf'(z)}{f(z)}+\frac{z\phi'(z)}{\phi(z)}+\gamma\prec R_{\beta+\gamma}(z)\quad\text{is equivalent to}$$

$$\beta\frac{zf'(z)}{f(z)}+J(\gamma,h)(z)\prec R_{\beta+\gamma}(z)$$

i.e. $f\in\mathcal{K}_{(\beta,\gamma),h}$ and using Lemma 1 we deduce that the function $F=A_{(\beta,\gamma),h}(f)$ is analytic in U. □

Theorem 2. *Let $\beta,\gamma\in\mathbf{C}$ with $\beta+\gamma>0$. For a function $h\in\mathcal{A}$ we denote by*

$$m=inf\left\{Re\,\gamma\frac{zh'(z)}{h(z)}:z\in U\right\}\quad and\ by\quad M=sup\left\{Re\,\gamma\frac{zh'(z)}{h(z)}:z\in U\right\}.$$

Let δ be a real number such that

$$m-(\beta+\gamma)<\delta\leq min\left\{m;m-\frac{\beta+\gamma-1}{2}\right\}\tag{3.1}$$

and

$$max\{0;M\}\leq\frac{\beta+\gamma}{{}_2F_1(1,2(\beta+\gamma+\delta-m),\beta+\gamma+1;\frac{1}{2})}\tag{3.2}$$

and suppose that h function satisfies the inequality

$$Re\,J(-\gamma,h)(z)\geq-\frac{1}{2}-\frac{\beta+\gamma}{{}_2F_1(1,2(\beta+\gamma+\delta-m),\beta+\gamma+1;\frac{1}{2})},\ z\in U.\tag{3.3}$$

If $f\in\mathcal{K}_{(\beta,\gamma),h}$ and Re $J(\beta,f)(z)>-\delta$, $z\in U$, then $F=A_{(\beta,\gamma),h}(f)$ given by (1.1) is univalent in U. In addition f is also univalent in U.

Proof. Since $f\in\mathcal{K}_{(\beta,\gamma),h}$, using Theorem 1 we have that $F=A_{(\beta,\gamma),h}(f)$ is analytic in U and by (1.1) we deduce

$$f(z)=\frac{1}{(\beta+\gamma)^{1/\beta}}F(z)\left[\beta\frac{h(z)}{h'(z)}\frac{F'(z)}{F(z)}+\gamma\right]^{1/\beta}.\tag{3.4}$$

Letting

$$L(z;t) = \frac{1}{(\beta+\gamma)^{1/\beta}} F(z) \left[(1+t)\beta \frac{h(z)}{h'(z)} \frac{F'(z)}{F(z)} + \gamma \right]^{1/\beta}, \ t \geq 0 \tag{3.5}$$

we will prove that $L(z;t)$ is a subordination chain. Using (3.5), a simple computation shows

$$z \frac{\partial L / \partial z}{\partial L / \partial t} = (1+t) T(z) + \gamma \frac{z h'(z)}{h(z)} \tag{3.6}$$

where

$$T(z) = J(\beta, F)(z) + \frac{z h'(z)}{h(z)} - \frac{z h''(z)}{h'(z)} - 1. \tag{3.7}$$

According to Lemma 3 and using (3.5), to prove that $L(z;t)$ is a subordination chain it is sufficient to show the next two inequalities:

$$Re\, T(z) > 0,\ z \in U \quad \text{and} \quad Re \left\{ T(z) + \gamma \frac{z h'(z)}{h(z)} \right\} > 0,\ z \in U. \tag{3.8}$$

From $f'(0) \neq 0$ we have $F'(0) \neq 0$ and if $L(z;t) = a_1(t) z + \cdots$ we get

$$\lim_{t \to +\infty} |a_1(t)| = \lim_{t \to +\infty} \left| \frac{\partial L(0;t)}{\partial z} \right| = \lim_{t \to +\infty} \left| \frac{F'(0)}{(\beta+\gamma)^{1/\beta}} (t\beta + \beta + \gamma)^{1/\beta} \right| = +\infty.$$

Since

$$J(\beta, f)(z) = p(z) + \frac{z p'(z)}{p(z)} - \gamma \frac{z h'(z)}{h(z)} \quad \text{where} \quad p(z) = T(z) + \gamma \frac{z h'(z)}{h(z)},$$

using the assumption $Re\, J(\beta, f)(z) > -\delta$, $z \in U$, we deduce

$$Re \left\{ p(z) + \frac{z p'(z)}{p(z)} \right\} > -\delta + m,\ z \in U. \tag{3.9}$$

Considering the differential equation

$$p(z) + \frac{z p'(z)}{p(z)} = J(\beta, f)(z) + \gamma \frac{z h'(z)}{h(z)},$$

from (3.1) we have

$$Re \left\{ J(\beta, f)(z) + \gamma \frac{z h'(z)}{h(z)} \right\} > -\delta + m \geq 0,\ z \in U$$

and by Lemma 5 we conclude that this equation has an analytic solution in U.

Denoting by $q(z) = \dfrac{p(z)}{\beta+\gamma}$, then $q(0) = 1$ and from (3.9) we obtain

$$Re \left\{ q(z) + \frac{z q'(z)}{(\beta+\gamma) q(z)} \right\} > \frac{m-\delta}{\beta+\gamma},\ z \in U. \tag{3.10}$$

Since (3.10) holds, by using Lemma 2 for $\beta := \beta + \gamma$, $\gamma := 0$, $\alpha := \dfrac{m-\delta}{\beta+\gamma}$ we obtain

$$Re\, q(z) > \frac{1}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})},\ z \in U$$

or

$$Re\, p(z) > \frac{\beta+\gamma}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})},\ z \in U. \tag{3.11}$$

From (3.11) and (3.2) we deduce

$$Re\, T(z) > \frac{\beta+\gamma}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})} - Re\, \gamma \frac{zh'(z)}{h(z)} >$$

$$> \frac{\beta+\gamma}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})} - M \geq 0,\ z \in U,$$

hence $Re\, T(z) > 0$, $z \in U$.

Similarly, from (3.11) and (3.2) we have

(3.12)

$$Re \left\{ T(z) + \gamma \frac{zh'(z)}{h(z)} \right\} > \frac{\beta+\gamma}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})} > 0,\ z \in U$$

hence both conditions of (3.8) are satisfied and according to Lemma 3, the function $L(z;t)$ is a subordination chain; thus $f(z) = L(z;0)$ is univalent in U.

Using (3.7) we obtain

$$J(\beta, F)(z) = T(z) + \frac{zh''(z)}{h'(z)} + 1 - \frac{zh'(z)}{h(z)} = T(z) + \gamma \frac{zh'(z)}{h(z)} + J(-\gamma, h)(z)$$

and by combining this equality with (3.12) and (3.3) we get

$$Re\, J(\beta, F)(z) > \frac{\beta+\gamma}{{}_2F_1(1, 2(\beta+\gamma+\delta-m), \beta+\gamma+1; \frac{1}{2})}$$

$$+ Re\, J(-\gamma, h)(z) \geq -\frac{1}{2},\ z \in U$$

hence by Lemma 4, the function F is univalent in U, which completes the proof of the theorem. □

4. Particular Cases

1. Taking $h(z) = ze^{\lambda z}$, $\lambda \leq 1$ in Theorem 2, for the case $\gamma \in \mathbf{R}$ we have $m = \gamma - |\gamma||\lambda|$ and $M = \gamma + |\gamma||\lambda|$. Then (3.3) becomes

$$Re\left[1-\gamma(1+\lambda z)-\frac{1}{1+\lambda z}\right]\geq$$

$$-\frac{1}{2}-\frac{\beta+\gamma}{{}_2F_1(1,2(\beta+\gamma+\delta-m),\beta+\gamma+1;\frac{1}{2})},\ z\in U. \tag{4.1}$$

Let consider the case $\beta+\gamma\geq 1$ and let $\delta:=\gamma-|\gamma||\lambda|-\dfrac{\beta+\gamma-1}{2}$. Then ${}_2F_1(1,2(\beta+\gamma+\delta-m),\beta+\gamma+1;\frac{1}{2})=2$ and (4.1) is equivalent to

$$Re\left[-\gamma\lambda z-\frac{1}{1+\lambda z}\right]\geq-\frac{3}{2}-\frac{\beta-\gamma}{2},\ z\in U. \tag{4.2}$$

Since

$$Re\left[-\gamma\lambda z-\frac{1}{1+\lambda z}\right]>-|\gamma||\lambda|-\frac{1}{1-|\lambda|},\ z\in U,$$

we deduce that (4.2) holds if $2|\gamma||\lambda|^2-(\beta-\gamma+2|\gamma|+3)|\lambda|+\beta-\gamma+1\geq 0$ and using Theorem 2 we obtain:

Corollary 1. *Let $\beta,\gamma\in\mathbf{R}$ with $\beta+\gamma\geq 1$ and let $\lambda\in\mathbf{C}$ with $|\lambda|\leq 1$ such that*

$$2|\gamma||\lambda|\leq\beta-\gamma$$

and

$$2|\gamma||\lambda|^2-(\beta-\gamma+2|\gamma|+3)|\lambda|+\beta-\gamma+1\geq 0.$$

If $f\in H(U)$ with $f(0)=0$, $f'(0)\neq 0$ and
$\beta\dfrac{zf'(z)}{f(z)}+1+\gamma(1+\lambda z)-\dfrac{1}{1+\lambda z}\prec R_{\beta+\gamma}(z)$ satisfies

$$Re\ J(\beta,f)(z)>|\gamma||\lambda|+\frac{\beta-\gamma-1}{2},\ z\in U$$

then $F=A_{(\beta,\gamma),ze^{\lambda z}}(f)$ is univalent in U; in addition f is also univalent in U.

2. Taking $h(z)=\dfrac{z}{1+\lambda z}$, $\lambda\leq 1$ in Theorem 2, a simple calculus shows that if $\gamma\geq 0$ then $m=\dfrac{\gamma}{1+|\lambda|}$ and $M=\dfrac{\gamma}{1-|\lambda|}$.

Let consider the case $\beta+\gamma\geq 1$ and let $\delta:=\dfrac{\gamma}{1+|\lambda|}-\dfrac{\beta+\gamma-1}{2}$. By similar reasons to the first particular case, (3.3) becomes

$$Re\left\{\frac{-\gamma-1}{1+\lambda z}+\frac{1-\lambda z}{1+\lambda z}\right\}\geq-\frac{1}{2}-\frac{\beta+\gamma}{2},\ z\in U$$

and since

$$Re\left\{\frac{-\gamma-1}{1+\lambda z}+\frac{1-\lambda z}{1+\lambda z}\right\}>\frac{-\gamma-1}{1-|\lambda|}+\frac{1-|\lambda|}{1+|\lambda|},\ z\in U,$$

by using Theorem 2 we have:

Corollary 2. *Let* $\beta, \gamma \in \mathbf{R}$ *with* $\gamma \geq 0$ *and* $\beta + \gamma \geq 1$ *and let* $\lambda \in \mathbf{C}$ *such that*

$$\frac{-\gamma - 1}{1 - |\lambda|} + \frac{1 - |\lambda|}{1 + |\lambda|} \geq -\frac{1}{2} - \frac{\beta + \gamma}{2}$$

and

$$|\lambda| \leq \frac{\beta - \gamma}{\beta + \gamma}.$$

If $f \in H(U)$ *with* $f(0) = 0$, $f'(0) \neq 0$ *and* $\beta \frac{zf'(z)}{f(z)} + \frac{\gamma - \lambda z}{1 + \lambda z} \prec R_{\beta+\gamma}(z)$ *satisfies*

$$Re\ J(\beta, f)(z) > \frac{\beta + \gamma - 1}{2} - \frac{\gamma}{1 + |\lambda|},\ z \in U$$

then $F = A_{(\beta,\gamma), \frac{z}{1+\lambda z}}(f)$ *is univalent in* U*; in addition* f *is also univalent in* U.

References

[1] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., 135(1969), 337-351

[2] R.J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16(1965), 755-758

[3] W.M. Causey and W.L. White, *Starlikeness of certain functions with integral representations*, J. Math. Anal. Appl., 64(1978), 458-466

[4] S.S. Miller and P.T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. Appl., 157,1(1991), 147-165

[5] S.S. Miller and P.T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations, 67(1987), 199-211

[6] S.S. Miller and P.T. Mocanu, *Integral operators on certain classes of analytic functions*, Univalent Functions, Fractional Calculus and their Applications, Halstead Press, J.Willey (1989), 153-166

[7] S.S. Miller, P.T. Mocanu and M.O. Reade, *Starlike integral operators*, Pacific J. Math., 79(1978), 157-168

[8] P.T. Mocanu, *Some integral operators and starlike functions*, Rev. Roum. Math. Pures Appl., 31(1986), 231-235

[9] P.T. Mocanu, D. Ripeanu and I. Şerb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj), 23(46)(1981), 225-230

[10] Ch. Pommerenke, *"Univalent Functions"*, Vanderhoeck and Ruprecht, Göttingen, 1975

[11] K. Sakaguchi, *A note on p-valent functions*, J. Math. Soc. Japan, 14(1962), 312-321

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