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THE EXTRARESOLVABILITY OF SOME FUNCTION SPACES*

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ABSTRACT. A space X is said to be *extraresolvable* if X contains a family $\mathcal{D}$ of dense subsets such that the intersection of every two elements of $\mathcal{D}$ is nowhere dense and $|\mathcal{D}| > \Delta(X)$, where $\Delta(X) = \min\{|U| : U$ is a nonempty open subset of $X\}$ is the ***dispersion character*** of X. In this paper, we study the extraresolvability of some function spaces $C_p(X)$ equipped with the pointwise convergence topology. We show that $C_p(X)$ is not extraresolvable provided that X satisfies one of the following conditions: X is metric; $nw(X) = \omega$; X is normal, $e(X) = nw(X)$ and either $e(X)$ is attained or $cf(e(X))$ is countable. Hence, $C_p(\mathbb{R})$ and $C_p(\mathbb{Q})$ are not extraresolvable. We establish the equivalences $2^{\omega} < 2^{\omega_1}$ iff $C_p([0, \omega_1))$ is extraresolvable; and, under GCH, for every infinite cardinal κ, the space $C_p([0, \kappa))$ is extraresolvable iff $cf(\kappa) > \omega$, where $[0, \kappa)$ has the order topology. We also prove that if $\kappa^{<cf(\kappa)} = \kappa$ and $cf(\kappa) > \omega$, then $C_p(\{0, 1\}^{\kappa})$ is extraresolvable; and that $C_p(\beta(\kappa))$ is extraresolvable, for every infinite cardinal κ with the discrete topology. It is shown that $C_p([0, \beta_{\omega_1}))$ is extraresolvable, where β_{ω_1} is the beth cardinal corresponding to ω_1. Under GCH, for a compact space X, we have that $cf(w(X)) > \omega$ iff $C_p(X)$ is extraresolvable. We proved that $2^{\omega} < 2^{\omega_1}$ is equivalent to the statement "$C_p(\{0, 1\}^{\omega_1})$ is strongly extraresolvable".

0. INTRODUCTION

All topological spaces considered in this paper are Tychonoff without isolated points. If X is a space, then $C_p(X)$ will denote the space of all real-valued continuous functions on X with the pointwise convergence topology (i. e., the topology on $C_p(X)$ inherited from the space $\mathbb{R}^X$). We allow X to have isolated points only when we deal with $C_p(X)$.

E. Hewitt [He] called a space *resolvable* if it has two disjoint dense subsets. The class of resolvable spaces is very extensive: E. Hewitt [He] proved that

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all metric spaces and all locally compact spaces are resolvable (see [CG1, Th. 3.7]). Years later, Ceder [Ce] introduced the spaces which contain k-many pairwise disjoint dense subsets, for a cardinal $\kappa \geq 2$, and he called them *κ-resolvable.* It is evident that a space X cannot be κ-resolvable for any $\kappa > \Delta(X)$. A $\Delta(X)$-resolvable space X is called *maximally resolvable.* Ceder [Ce] showed that metric spaces and locally compact spaces are maximally resolvable (see [CG1]). The following class of spaces, which contains the class of ω-resolvable spaces, was investigated by V. I. Malykhin [Ma].

Definition 0.1. (Malykhin) *A space X is called extraresolvable if there exists a family $\mathcal{D}$ of dense subsets of X such that $\mid \mathcal{D} \mid > \Delta(X)$ and $D \cap E$ is nowhere dense whenever $D, E \in \mathcal{D}$ and $D \neq E$.*

The authors of [CG2] and [CG3] slightly generalize Malykhin's extraresolvability as follows.

Definition 0.2. *A space X is called strongly extraresolvable if there exists a family $\mathcal{D}$ of dense subsets of X such that $\mid \mathcal{D} \mid > \Delta(X)$ and $|D \cap E| < nwd(X)$ whenever $D, E \in \mathcal{D}$ and $D \neq E$, where $nwd(X)$ is the nowhere density number of X defined by $nwd(X) = \min\{|A| : A \subseteq X, A$ is not nowhere dense in $X\}$.*

Notice that every strongly extraresolvable space is extraresolvable and the reader may find examples of extraresolvable spaces which are not strongly extraresolvable in [CG2]. We know that the rational numbers $\mathbb{Q}$ is extraresolvable, and that the real line $\mathbb{R}$ cannot be extraresolvable (for stronger results see [GMT]). Some other topological properties of extraresolvable spaces, not considered here, are available in [AGT], [GMT], [CG2] and [CG3].

For every space X, the space of all bounded real-valued continuous functions on X, denoted by $C^*(X)$, considered as a subspace of $C(X)$ is a metric space (see [GJ]). Hence, by Ceder's Theorem quoted above, $C^*(X)$ is maximally resolvable, for every space X. Since every open subset of $C_p(X)$ contains a topological copy of $C^*(X)$ and $|C^*(X)| = |C_p(X)|$, we have that $\Delta(C^*(X)) = \Delta(C_p(X)) = |C_p(X)|$ and, by Theorem 2.2 of [CG1], $C_p(X)$ is maximally resolvable for every space X (this fact was noticed by V. V. Tkachuk). This remark suggests the question: When is $C_p(X)$ extraresolvable, for a space X ?. It turns out that the answer is, in some cases, independent from the axioms of ZFC.

In the present paper, the first Section contains some preliminary results, and, in the second Section, we study the extraresolvability of some function spaces and show, in ZFC, that many of the $C_p(X)$ spaces are not extraresolvable. In this Section, we also give an Example of a space X for which $C_p(X)$ is strongly extraresolvable.

1. Preliminaries

Cardinal variables are denoted by the Greek letters α, κ and λ. For a set X and an infinite cardinal number α, we put $[X]^\alpha = \{A \subseteq X : |A| = \alpha\}$, the meaning of $[X]^{<\alpha}$ and $[X]^{\leq\alpha}$ should be clear. We use the standard notation (for definitions see [Ar]) $d(X)$, $e(X)$, $t(X)$, $c(X)$, $\pi w(X)$, $nw(X)$, $iw(X)$ and $w(X)$ for the density, the extent, the tightness, the cellularity, the π-weight, the net weight, the i-weight and the weight of a space X, respectively. A family $\mathcal{N}$ of subsets of a space X is said to be a *π-network* if $\emptyset \notin \mathcal{N}$ and each nonempty open subset of X contains an element of $\mathcal{N}$. If $\mathcal{A}$ is a family of nonempty subsets of a set X, then $\Delta(\mathcal{A}) = \min\{|A| : A \in \mathcal{A}\}$.

Several classes of spaces are contained in the class of maximally resolvable spaces: E. G. Pytkeev [Py] proved that k-spaces are maximally resolvable and V. I. Malykhin and I. V. Protasov [MP] showed that totally bounded groups are maximally resolvable: For background and references about maximal resolvability the reader is referred to [CG1]. Here, we only state the results of E. G. Pytkeev and A. G. El'kin:

Theorem 1.1. *Let X be a space.*

1. [**A. G. El'kin** [El]] *If $\pi w(X) \leq \Delta(X)$, then X is maximally resolvable.*
2. [**E. G. Pytkeev** [Py]] *If $t(X) < \Delta(X)$, then X is maximally resolvable.*

In the next Theorem, we establish the connection between extraresolvable and ω-resolvable spaces and give a condition to see when a space is not extraresolvable (the proofs are available [GMT]).

Theorem 1.2. *Let X be a space. Then,*

1. *if X is extraresolvable, then X is ω-resolvable; and*
2. *if $|X|^{nw(X)} = \Delta(X)$, then X is not extraresolvable. Furthermore, if there is a π-base of size at most $\Delta(X)$, then X is maximally resolvable.*

We remark from Theorem 1.2 that X^κ is never extraresolvable for every space X and for every cardinal $\kappa \geq nw(X)$. Hence, compact topological groups and compact metric spaces fail to be extraresolvable.

The next conditions guarantee the strong extraresolvability of some spaces (these conditions are taken from [AGT]).

Lemma 1.3. *Let κ and α be cardinal numbers with $\omega \leq \alpha$. If X satisfies one of the following two lists of conditions:*

1. $cf(\kappa) = \alpha$;
2. *X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \kappa \leq \kappa^{<\alpha} \leq \Delta(\mathcal{N})$;*
3. *every subset of X of size $< \kappa$ is nowhere dense; and*
4. $\Delta(X) < \kappa^\alpha$,

or

1'. *X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \alpha \leq \kappa^{<\alpha} \leq \Delta(\mathcal{N})$;*
2'. *every subset of X of size $< \alpha$ is nowhere dense; and*
3'. *$\Delta(X) < \kappa^{\alpha}$,*

then X is strongly extraresolvable.

Using Lemma 1.3, it is shown in [AGT] that the assumption $2^{\kappa} < 2^{\kappa^+}$ is equivalent to the extraresolvability of the power $(\kappa^+)^{\kappa}$, for every infinite cardinal number κ. The authors of [AGT] also noticed that many powers, although several of them are not extraresolvable, admit an extraresolvable dense subspace.

The proof of the next result resembles the one of Theorem 3.1 of [GMT].

Lemma 1.4. *If X satisfies that*

$$t(X) < d(X) = nwd(X) \leq d(X)^{<cf(d(X))} \leq \Delta(X) < d(X)^{cf(d(X))},$$

then X is strongly extraresolvable.

Proof: By Theorem 1.1, X is maximally resolvable and hence we may find a pairwise disjoint family $\mathcal{D}$ of dense subsets of X with $|\mathcal{D}| = d(X)^{<cf(d(X))}$. Let D be a dense subset of X with $|D| = d(X)$. We faithfully index $\mathcal{D}$ and D as $\{D_{\nu,s} : \nu < d(X), s \in d(X)^{<cf(d(X))}\}$ and $\{d_\nu : \nu < d(X)\}$, respectively. For each $\nu < d(X)$ and each $s \in d(X)^{<cf(d(X))}$, we choose $N_{\nu,s} \in [D_{\nu,s}]^{\leq t(X)}$ such that $d_\nu \in cl_X N_{\nu,s}$. Now, for $f \in d(X)^{cf(d(X))}$ we define

$$N_f = \bigcup_{\nu<d(X)} \bigcup_{\theta<cf(d(X))} N_{\nu,f|_\theta}.$$

It is not hard to see that N_f is dense in X for each $f \in d(X)^{cf(d(X))}$. If $f, g \in d(X)^{cf(d(X))}$ and $f \neq g$, then $|N_f \cap N_g| < d(X) = nwd(X)$. This shows that X is strongly extraresolvable. □

Two important consequences of Lemma 1.4 are the following.

Theorem 1.5. *If X satisfies that*

1. *$t(X) < d(X) \leq d(X)^{<cf(d(X))} \leq \Delta(X) < d(X)^{cf(d(X))}$; and*
2. *$d(V) = d(X)$, for every non-empty open subset V of X,*

then X is strongly extraresolvable.

Proof: By Lemma 1.4, it suffices to show that $d(X) = nwd(X)$. We always have that $d(X) \geq nwd(X)$. Let $A \in [X]^{<d(X)}$ and suppose that there is a nonempty open subset V of X with $V \subseteq cl_X A$. Then, $V \cap A$ is dense in V, but this is impossible since $|A| < d(X) = d(V)$. Therefore, $d(X) \leq nwd(X)$ □

Theorem 1.6. *If G is a topological group such that*

1. *$t(G) < d(G) \leq d(G)^{<cf(d(G))} \leq \Delta(G) < d(G)^{cf(d(G))}$ and*
2. *$c(G) < d(G)$,*

then G is strongly extraresolvable.

Proof: In virtue of Lemma 1.4, we only need to verify that $d(G) \leq nwd(G)$. Let $A \in [G]^{<d(G)}$. Without loss of generality, we may assume that A is a subgroup of G. Then, $H = cl_G A$ is also a subgroup of G. Let us suppose that H has non-void interior. Then, H is a clopen subgroup of G and since $|G/H| \leq c(G)$ and $d(H) \leq |A|$, we have that $d(G) \leq c(G) \cdot |A| < d(G)$, which is a contradiction. So, $d(G) \leq nwd(G)$ and hence $d(G) = nwd(G)$. □

The following Lemma is taken from [GMT].

Lemma 1.7. *If X satisfies that*

$$t(X) < d(X) \leq \Delta(X) < 2^{d(X)}$$

and $d(X)$ is a strong limit cardinal, then X is strongly extraresolvable.

Lemma 1.8. *If X satisfies that*

$$t(X) < d(X) = nwd(X) = \Delta(X)$$

then X is strongly extraresolvable.

Proof: From Theorem 1.1 we obtain that X is maximally resolvable. Hence, there is a pairwise disjoint family $\{D_\nu : \nu < d(X)\}$ of dense subsets of X. Fix a dense subset $D = \{d_\nu : \nu < d(X)\}$ of X with $|D| = d(X)$. Let $\{(\nu_\xi, \mu_\xi) : \xi < d(X)\}$ be an indexing of $d(X) \times d(X)$. Now, choose $S_{(\nu_0,\mu_0)} \subseteq D_{\nu_0}$ such that $|S_{(\nu_0,\mu_0)}| \leq t(X)$ and $d_{\nu_0} \in cl_X S_{(\nu_0,\mu_0)}$. By transfinite induction, for every $\xi < d(X)$ we may find $S_{(\nu_\xi,\mu_\xi)} \subseteq D_{\nu_\xi}$ such that

1. $|S_{(\nu_\xi,\mu_\xi)}| \leq t(X)$;
2. $d_{\nu_\xi} \in cl_X S_{(\nu_\xi,\mu_\xi)}$; and
3. $S_{(\nu_\xi,\mu_\xi)} \subseteq D_{\nu_\xi} - (\bigcup_{\zeta<\xi} S_{(\nu_\zeta,\mu_\zeta)})$.

Abandoning earlier enumeration, we have defined a pairwise disjoint family $\{S_{(\xi,\zeta)} : (\xi,\zeta) \in d(X) \times d(X)\}$ of subsets of X which satisfies the following:

1. $|S_{(\xi,\zeta)}| \leq t(X)$, for all $(\xi,\zeta) \in d(X) \times d(X)$;
2. $d_\xi \in cl_X S_{(\xi,\zeta)}$, for all $(\xi,\zeta) \in d(X) \times d(X)$; and
3. if $(\xi,\zeta) \neq (\nu,\mu)$, then $S_{(\xi,\zeta)} \cap S_{(\nu,\mu)} = \emptyset$, for $(\xi,\zeta),(\nu,\mu) \in d(X) \times d(X)$.

By Lemma 12.8 of [GJ], there is a family $\mathcal{F} \subseteq d(X)^{d(X)}$ such that $|\mathcal{F}| > d(X)$ and $|\{\xi < d(X) : f(\xi) = g(\xi)\}| < d(X)$ whenever $f, g \in \mathcal{F}$ and $f \neq g$. For each $f \in \mathcal{F}$ we define

$$N_f = \bigcup_{\xi < d(X)} S_{(\xi, f(\xi))}.$$

We then have that N_f is dense in X for each $f \in \mathcal{F}$, and if $f, g \in \mathcal{F}$ are distinct, then $|N_f \cap N_g| < d(X) = nwd(X)$. Therefore, X is strongly extraresolvable. □

It is shown in [CG2] that if $w(X) \leq \Delta(X)$, then there is a family $\mathcal{E}$ of dense subsets of X such that $|D \cap E| < \Delta(X)$ for distinct $D, F \in \mathcal{E}$. But, some spaces with this property are far of being extraresolvable (for instance, the real line $\mathbb{R}$). By adding the obvious extra condition, we get extraresolvability:

Lemma 1.9. [CG2] *If X satisfies that*

$$w(X) = \Delta(X) = nwd(X),$$

then X is strongly extraresolvable.

In the next Lemma, we list some properties of the function spaces that we shall use in the second Section (the proofs are available in [Ar], [McN], [No] and [CH]).

Proposition 1.10. *1. For every space X, we have the following.*

a. $|C_p(X)|^\omega = |C_p(X)| = w(\beta(X))^\omega$;
b. $\Delta(C_p(X)) = |C_p(X)| \leq 2^{d(X)}$;
c. $d(C_p(X)) = iw(X)$;
d. $w(C_p(X)) = |X|$;
e. $c(C_p(X)) = \omega$; *and*
f. $nw(C_p(X)) = nw(X)$.

2. If X is a metric space, then $|C_p(X)| = 2^{d(X)}$.

3. If κ is an infinite cardinal, then

a. $|C_p([0,\kappa))| = \kappa^\omega = \Delta(C_p([0,\kappa)))$ *and*
b. $w(C_p([0,\kappa))) = \kappa$.

4. If X is compact, then $t(C_p(X)) = \omega$.

Next, we shall show that $d(C_p(X)) = nwd(C_p(X))$, for every space X. First, we give some notation and prove a Lemma.

If I is an open interval of $\mathbb{R}$ and $x \in X$, then

$$[x, I] = \{f \in C_p(X) : f(x) \in I\}$$

will denote a subbasic open set of $C_p(X)$.

Lemma 1.11. *Let X be a space. If V is a basic open set of $C_p(X)$, then*

$$iw(X) \leq d(V).$$

Proof: Let $V = \bigcap_{j \leq n}[x_j, I_j]$ be a basic open set of $C_p(X)$, where $n < \omega$, and let D be a dense subset of V having cardinality $d(V)$. We claim that D separates poins of X. Indeed, fix two distinct points $x, y \in X$. We consider four cases.

Case I. $\{x,y\} \cap \{x_0, ..., x_n\} = \emptyset$. Since D is dense in V there is $f \in V \cap [x,(0,1)] \cap [y,(1,2)] \cap D$; hence, $f(x) \neq f(y)$ and $f \in D$.

Case II. $\{x,y\} \subseteq \{x_0, ..., x_n\} = \emptyset$. Without loss of generality, we may assume that $x = x_0$ and $y = x_1$. Choose two disjoint open intervals J_0 and J_1 so that $J_0 \subseteq I_0$ and $J_1 \subseteq I_1$. We may find $f \in V \cap [x_0, J_0] \cap [x_1, J_1] \cap D$. Then, we have that $f(x_0) \neq f(x_1)$ and $f \in D$.

The cases when $x \in \{x_0, ..., x_n\}$ and $y \notin \{x_0, ..., x_n\}$ and when $y \in \{x_0, ..., x_n\}$ and $x \notin \{x_0, ..., x_n\}$ are left to the reader. This shows our claim. Thus, we have that D separates points of X. It follows that the evaluation map $e : X \to \mathbb{R}^D$, given by $e(x)(f) = f(x)$ for each $x \in X$ and for each $f \in D$, is one-to-one (see [Ar] or [McN]). By definition, $iw(X) \leq |D| = d(V)$. □

Theorem 1.12. *For every space X, we have that*

$$d(C_p(X)) = nwd(C_p(X)).$$

Proof: In Proposition 1.10, we pointed out that $iw(X) = d(C_p(X))$ (a proof of this fact lies in [Ar] and [McN]), and it is evident that $nwd(Y) \leq d(Y)$ for every space Y. So, it is enough to show that $iw(X) \leq nwd(C_p(X))$. In fact, let $A \subseteq C_p(X)$ with $|A| < iw(X)$. Assume that there is a basic open set V of $C_p(X)$ such that $V \subseteq cl_{C_p(X)}A$. Since $V \cap A$ is a dense subset of V, by Lemma 1.11, we must have that $iw(X) \leq d(V) \leq |V \cap A| \leq |A|$, which is impossible. Therefore, $iw(X) \leq nwd(C_p(X))$ as required. □

Another proof of Theorem 1.12 lies in [CG2].

2. Spaces of continuous functions

In the paper [Ma], V. I. Malykhin showed that every dense in itself countable subspace of $C_p(X)$ is extraresolvable whenever X is an analytic set from a compact space; for instance, the irrational numbers. On the other hand, if κ is an infinite cardinal number equipped with the discrete topology, then $C_p(\kappa) = \mathbb{R}^\kappa$ cannot be extraresolvable, by Theorem 1.2. In a more general setting, we have:

Lemma 2.1. *If X is a space such that $|C_p(X)|^{nw(X)} = |C_p(X)|$, then $C_p(X)$ is not extraresolvable.*

Proof: By hypothesis, we have that

$$|C_p(X)|^{nw(C_p(X))} = |C_p(X)|^{nw(X)} = |C_p(X)| = \Delta(C_p(X)),$$

since $nw(C_p(X)) = nw(X)$, by Proposition 1.10. So, by Theorem 1.2, $C_p(X)$ cannot be extraresolvable. □

It is possible to show, in ZFC, that $C_p(X)$ is not extraresolvable for several of the familiar spaces $X's$.

Theorem 2.2. *The space $C_p(X)$ is not extraresolvable provided that X satisfies one of the following conditions:*

1. *X is metric;*
2. *$nw(X) = \omega$;*
3. *X is normal, $e(X) = nw(X)$ and either $e(X)$ is attained or $cf(e(X))$ is countable.*

Proof: In all cases, we shall prove that $|C_p(X)|^{nw(X)} = |C_p(X)|$ and then apply Lemma 2.1.

1. X is metric. By Proposition 6.1 of [CH], $|C_p(X)| = 2^{d(X)}$ and since $d(X) = nw(X)$ (see [Ho, 8.1]), we must have that

$$|C_p(X)|^{nw(X)} = |C_p(X)|^{d(X)} = 2^{d(X)} = |C_p(X)|.$$

2. $nw(X) = \omega$. From Proposition 1.10 we obtain that

$$|C_p(X)|^{\omega} = |C_p(X)| = |C_p(X)|^{nw(X)}.$$

3. X is normal and $e(X) = nw(X)$. First, suppose that $e(X)$ is attained. Then, $2^{e(X)} \leq |C_p(X)|$ and hence $2^{e(X)} = 2^{nw(X)} \leq |C_p(X)| \leq 2^{d(X)} \leq 2^{nw(X)}$. Thus, $|C_p(X)|^{nw(X)} = 2^{nw(X)} = |C_p(X)|$. Now, assume that $cf(e(X)) = \omega$. Let $\{\kappa_n : n < \omega\}$ be a strictly increasing and cofinal sequence of cardinal numbers of $e(X)$. Since $2^{\kappa_n} \leq |C_p(X)|$ for every $n < \omega$, we have that $2^{e(X)} = \prod_{n<\omega} 2^{\kappa_n} \leq |C_p(X)|^{\omega} = |C_p(X)| \leq 2^{nw(X)}$. Hence, $2^{e(X)} = 2^{nw(X)} = |C_p(X)| = |C_p(X)|^{nw(X)}$. □

Theorem 2.2 implies that $C_p(\mathbb{R})$, $C_p(\mathbb{Q})$, $C_p([0,1])$, $C_p(\mathbb{I})$, where $\mathbb{I}$ is the space of irrational numbers, and $C_p(X)$, for every countable space X, are not extraresolvable.

Next, we will see that, under GCH, $C_p([0,\kappa))$ is extraresolvable for every cardinal κ with $cf(\kappa) \geq \omega_1$.

Lemma 2.3. *For every uncountable cardinal κ, we have that*

$$nwd(C_p([0,\kappa))) = \kappa,$$

where $[0,\kappa)$ has the order topology.

Proof: By Proposition 1.10 and Theorem 1.12, we have

$$nwd(C_p([0,\kappa))) = d(C_p([0,\kappa))) = iw([0,\kappa)) \leq \kappa.$$

Fix an infinite cardinal λ smaller than κ. Then there exists a regular cardinal ρ such that $\lambda < \rho \leq \kappa$. Let $F \in [C_p([0,\kappa))]^{\lambda}$. We may find $\theta < \rho$ such that $f|_{[\theta,\rho)}$ is constant for every $f \in F$. We claim that if $g \in cl_{C_p([0,\kappa))}F$, then $g|_{[\theta,\rho)}$ is also constant. In fact, let $g \in cl_{C_p([0,\kappa))}F$ and suppose that $g(\theta) \neq g(\nu)$ for some $\theta < \nu < \rho$. Choose two open disjoint subsets V and U of $\mathbb{R}$ so that $g(\theta) \in V$ and $g(\nu) \in U$. If $W = \pi_{\theta}^{-1}(V) \cap \pi_{\nu}^{1}(U)$, where $\pi_{\theta} : \mathbb{R}^{[0,\kappa)} \to \mathbb{R}$ and $\pi_{\nu} : \mathbb{R}^{[0,\kappa)} \to \mathbb{R}$ are the projection maps, then W is an open neighborhood of g and $W \cap F = \emptyset$, which is a contradiction. This shows our claim. Thus,

$$cl_{C_p([0,\kappa))}F \subseteq \bigcup_{r\in\mathbb{R}} (\mathbb{R}^{[0,\theta)} \times \{r\}^{[\theta,\rho)} \times \mathbb{R}^{[\rho,\kappa)})$$

and so $cl_{C_p([0,\kappa))}F$ has void interior. Therefore, $nwd(C_p([0,\kappa))) = \kappa$. □

Lemma 2.4. *Let κ be an uncountable cardinal. If $C_p([0,\kappa))$ is extraresolvable, then $\kappa^\omega < 2^\kappa$.*

Proof: If $\kappa^\omega = 2^\kappa$, then $|C_p([0,\kappa))|^{w(C_p([0,\kappa)))} = (\kappa^\omega)^\kappa = 2^\kappa = \kappa^\omega = |C_p([0,\kappa))| = \Delta(C_p([0,\kappa)))$, and by Lemma 2.1, $C_p([0,\kappa))$ would not be extraresolvable. Therefore, $\kappa^\omega < 2^\kappa$. □

Lemma 2.5. *If $\kappa = \kappa^\omega$, then $C_p([0,\kappa))$ is strongly extraresolvable.*

Proof: We have that

$$w(C_p([0,\kappa))) = \kappa = \kappa^\omega = \Delta(C_p([0,\kappa)))$$

and, by Lemma 2.3, $nwd(C_p([0,\kappa))) = \kappa = \kappa^\omega = \Delta(C_p([0,\kappa)))$. So, by Lemma 1.9, $C_p([0,\kappa))$ is strongly extraresolvable. □

We define $\beta_0 = \omega$, $\beta_{\theta+1} = 2^{\beta_\theta}$, for every ordinal θ, and if θ is a limit ordinal, then $\beta_\theta = \sum_{\nu<\theta} \beta_\nu$. In ZFC, the space $C_p([0,\beta_{\omega_1}))$ is strongly extraresolvable because of Lemma 2.5.

Two consequences of Lemmas 2.4 and 2.5 are the following.

Theorem 2.6. [GCH] *For an infnite cardinal κ, the following are equivalent.*

1. *$C_p([0,\kappa))$ is strongly extraresolvable;*
2. *$C_p([0,\kappa))$ is extraresolvable;*
3. *$cf(\kappa) \geq \omega_1$;*
4. *$[0,\kappa)$ is pseudocompact.*

Proof: The implication (1) ⇒ (2) is evident and the equivalence (3) ⇔ (4) is well-known (see [GJ]).

(2) ⇒ (3). Assume GCH. By Lemma 2.4, $\kappa^\omega < 2^\kappa = \kappa^+$ and so $\kappa = \kappa^\omega$. Hence, $cf(\kappa) > \omega$.

(3) ⇒ (1). GCH implies that $\kappa = \kappa^\omega = \kappa^{<cf(\kappa)}$, since $cf(\kappa) \geq \omega_1$. By Lemma 2.5, $C_p([0,\kappa))$ is strongly extraresolvable. □

Theorem 2.7. *The following are equivalent.*

1. $2^\omega < 2^{\omega_1}$;
2. *$C_p([0,\omega_1))$ is strongly extraresolvable;*
3. *$C_p([0,\omega_1))$ is extraresolvable.*

Since $2^\omega = 2^{\omega_1}$ and $2^\omega < 2^{\omega_1}$ are consistent with the axioms of ZFC, by Theorem 2.7, the extraresolvability of $C_p([0,\omega_1))$ is independent from the axioms of ZFC.

Corollary 2.8. [GCH] *The space $C_p([0,\kappa))$ is not extraresolvable for every cardinal κ with $cf(\kappa) = \omega$.*

Proof: Assume GCH. Let κ be a cardinal with $cf(\kappa) = \omega$. Then,

$$|C_p([0,\kappa))| = \kappa^\omega = \kappa^+ = (\kappa^+)^\kappa = |C_p([0,\kappa))|^{nw([0,\kappa))}.$$

By Lemma 2.1, $C_p([0,\kappa))$ is not extraresolvable. □

Now, we shall study the extraresolvability of a $C_p(X)$ space for the case when X is compact. For the next result we need some notation and a concept: If $\emptyset \neq T \subseteq \kappa$, then $\pi_T : X^\kappa \to X^T$ will denote the projection map on X^T, and we say that a function $f : X^\kappa \to Y$ *depends on* $\leq \lambda$ *coordinates* if there is $J \in [\kappa]^{\leq\lambda}$ such that $f(x) = f(y)$ whenever $\pi_J(x) = \pi_J(y)$ and $x, y \in X^\kappa$.

Lemma 2.9. *Let $\kappa > \omega$ be a cardinal number and let X be a space with $d(X) < \kappa$. Then every subset of $C_p(X^\kappa)$ of size $< \kappa$ is nowhere dense.*

Proof: Let $F = \{f_\xi : \xi < \lambda\} \subseteq C_p(X^\kappa)$ with $\lambda < \kappa$. By Theorem 10.14 of [CN1], for every $\xi < \lambda$, there is $S_\xi \in [\kappa]^{\leq d(X)}$ such that f_ξ depends on S_ξ. Put $S = \bigcup_{\xi<\lambda} S_\xi$ and notice that $|S| \leq d(X) \cdot \lambda < \kappa$. Fix $f \in cl_{C_p(X^\kappa)}F$. Suppose that there are $x, y \in X^\kappa$ such that $\pi_S(x) = \pi_S(y)$ and $f(x) \neq f(y)$. Then, there are disjoint open subsets U and V of $\mathbb{R}$ with $f(x) \in U$ and $f(y) \in V$. Let $W = \{g \in C_p(X^\kappa) : g(x) \in U$ and $g(y) \in V\}$. We have that W is an open neighborhood of f in $C_p(X^\kappa)$. So, there is $\xi < \lambda$ such that $f_\xi \in W$, but this is a contradiction since $f_\xi(x) \neq f_\xi(y)$ and $\pi_S(x) = \pi_S(y)$. This shows that every element of $cl_{C_p(X^\kappa)}F$ depends on S. It is not hard to see that this last condition implies that $cl_{C_p(X^\kappa)}F$ has void interior. □

Theorem 2.10. *If $\kappa^{<cf(\kappa)} = \kappa$ and $cf(\kappa) > \omega$, then $C_p(X^\kappa)$ is strongly extraresolvable for every compact space X with $w(X) \leq \kappa$ and $d(X) < \kappa$.*

Proof: Let X be a compact space with $w(X) \leq \kappa$ and $d(X) < \kappa$. By Proposition 1.10, $|C_p(X^\kappa)| = w(X^\kappa)^\omega = \kappa^\omega$. Then, applying again Proposition 1.10,

$$nw(C_p(X^\kappa)) = nw(X^\kappa) \leq \kappa^\omega = \kappa = \kappa^{<cf(\kappa)} = |C_p(X^\kappa)| = \Delta(C_p(X^\kappa)) < \kappa^{cf(\kappa)}.$$

According to Lemma 2.9, every subset of $C_p(X^\kappa)$ of size $< \kappa$ is nowhere dense. So, by Lemma 1.3, $C_p(X^\kappa)$ is strongly extraresolvable. □

Corollary 2.11. *If $\kappa^{<cf(\kappa)} = \kappa$ and $cf(\kappa) > \omega$, then $C_p(\{0,1\}^\kappa)$ is strongly extraresolvable.*

Thus, the space $C_p(\{0,1\}^{\beta_{\omega_1}})$ is strongly extraresolvable.

Theorem 2.12. *If X is a compact space such that*

$$\omega < w(X) \leq w(X)^\omega = w(X)^{<cf(w(X))} < w(X)^{cf(w(X))},$$

then $C_p(X)$ is strongly extraresolvable. In particular, under GCH, $C_p(X)$ is strongly extraresolvable provided that X is a compact spaces with $cf(w(X)) > \omega$.

Proof: According to Proposition 1.10, $t(C_p(X)) = c(C_p(X)) = \omega$ and since X is compact, we have that $\Delta(C_p(X)) = w(X)^\omega$ and $d(C_p(X)) = iw(X) = w(X)$. If we replace $d(C_p(X))$ by $w(X)$ in the inequality of the hypothesis, then

$$\omega = t(C_p(X)) = c(C_p(X)) < d(C_p(X)) \leq d(C_p(X))^{<cf(d(C_p(X)))}$$
$$= \Delta(C_p(X)) < d(C_p(X))^{cf(d(C_p(X)))}.$$

The conclusion now follows from Theorem 1.6. It is clear that GCH implies $w(X) = w(X)^\omega = w(X)^{<cf(w(X))} < w(X)^{cf(w(X))}$, for every compact space X with $cf(w(X)) > \omega$. Now, we apply the first part of the Theorem. □

Thus, we have that $2^\omega < 2^{\omega_1}$ implies that $C_p(\{0,1\}^{\omega_1})$ is strongly extraresolvable.

Theorem 2.13. *If X is a compact space and $w(X)$ is a strong limit cardinal with $w(X)^\omega < 2^{w(X)}$, then $C_p(X)$ is strongly extraresolvable.*

Proof: It follows from Proposition 1.10 that $t(C_p(X)) = \omega$. By compactness, we have that $d(C_p(X)) = iw(X) = w(X)$ and $\Delta(C_p(X)) = |C_p(X)| = w(X)^\omega$. Thus,

$$\omega = t(C_p(X)) < d(C_p(X)) = w(X) \leq w(X)^\omega$$
$$= \Delta(C_p(X)) < 2^{d(C_p(X))} = 2^{w(X)}.$$

In virtue of Lemma 1.7, $C_p(X)$ is extrongly extraresolvable. □

Theorem 2.14. *If X is compact and $w(X)^\omega = w(X)$, then $C_p(X)$ is strongly extraresolvable. Hence, $C_p(\beta(\kappa))$ is strongly extraresolvable, for every infinite cardinal κ equipped with the discrete topology.*

Proof: By Proposition 1.10 and Theorem 1.12, we have that

$$\omega = t(C_p(X)) < w(X) = d(C_p(X)) = w(X)^\omega$$
$$= \Delta(C_p(X)) = nwd(C_p(X)).$$

Now, we apply Lemma 1.8 to obtain that $C_p(X)$ is strongly extraresolvable. If κ is an infinite cardinal, then we have $w(\beta(\kappa)) = 2^\kappa$ and hence $w(\beta(\kappa))^\omega = w(\beta(\kappa))$. So, $C_p(\beta(\kappa))$ is strongly extraresolvable, for every infinite cardinal κ. □

Theorem 2.15. *If X is compact and $w(X)^\omega = 2^{w(X)}$, then $C_p(X)$ is not extraresolvable.*

Proof: Since X is compact, by Proposition 1.10, $\Delta(C_p(X)) = |C_p(X)| = w(X)^\omega = 2^{w(X)}$; hence, $|C_p(X)|^{nw(X)} = 2^{w(X)\cdot nw(X)} = 2^{w(X)} = |C_p(X)|$. By Lemma 2.1, $C_p(X)$ is not extraresolvable. □

We observe from Theorem 2.15 that if X is compact and $C_p(X)$ is extraresolvable, then $w(X)^\omega < 2^{w(X)}$. Let us consider the beth cardinal β_ω.

This cardinal number satisfies that $(\beta_\omega)^\omega = 2^{\beta_\omega}$ and is a strong limit cardinal. Thus, by Theorem 2.15, we have that $C_p(\{0,1\}^{\beta_\omega})$ is not extraresolvable. Also, notice that $C_p(\{0,1\}^\omega)$ is not extraresolvable, by Theorem 2.2. Under GCH, we have two very nice equivalences which follow directly from Theorems 2.12 and 2.15:

Corollary 2.16. [GCH] *For a compact space X, the following are equivalent.*

1. $cf(w(X)) = \omega$.
2. *$C_p(X)$ is not extraresolvable.*

Corollary 2.17. *For a compact space X, the following are equivalent.*

1. $2^\omega < 2^{\omega_1}$;
2. *$C_p(\{0,1\}^{\omega_1})$ is strongly extraresolvable;*
3. *$C_p(\{0,1\}^{\omega_1})$ is extraresolvable;*

Corollary 2.18. *Assuming GCH, the space $C_p(\{0,1\}^\kappa)$ is not extraresolvable for every cardinal κ with $cf(\kappa) = \omega$.*

Theorem 2.19. *Let κ be an infinite cardinal. If $2^\omega = 2^\kappa$, then $C_p(X)$ is not extraresolvable for every space X with $nw(X) \leq \kappa$.*

Proof: Assume $2^\omega = 2^\kappa$. Let X be space with $nw(X) \leq \kappa$. Then, $2^\omega \leq |C_p(X)| \leq 2^{d(X)} \leq 2^{nw(X)} \leq 2^\kappa$; hence,

$$2^\omega = 2^{nw(X)} = |C_p(X)| = |C_p(X)|^{nw(X)}.$$

By Lemma 2.1, we conclude that $C_p(X)$ is not extraresolvable. □

We finish the paper with some Problems and Questions that the authors were unable to solve.

Question 2.20. *For every X, does $C_p(X)$ have a dense extraresolvable subspace?*

Question 2.21. *Is there a space X such that $C_p(X)$ is extraresolvable and is not strongly extraresolvable?*

Problem 2.22. *Find a space X such that $C_p(X)$ is extraresolvable and $C_p(\beta(X))$ is not extraresolvable.*

By Theorem 2.14, the space $C_p(\beta(\kappa))$ is strongly extraresolvable and $C_p(\kappa) = \mathbb{R}^\kappa$ is not extraresolvable, for each infinite cardinal κ.

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