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SYMMETRIC (100,45,20)-DESIGNS WITH $E_{25} \cdot S_3$ AS FULL AUTOMORPHISM GROUP

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ABSTRACT. The construction of eight nonisomorphic new symmetric (100,45,20)-designs, having $E_{25} \cdot S_3$ as their full automorphism group, is presented.

1. PRELIMINARIES

Symmetric (100,45,20)-designs belong to Menon series consisting of all symmetric designs with parameters $(4t^2, 2t^2 - t, t^2 - t)$. The existence of such a design, on the basis of its equivalence with the existence of regular Hadamard matrix of order 100, has been known for a rather long time (see [3]). However, few constructions have been made so far ([2], [5]). Here we perform a construction of designs with given parameters making use of their tactical decomposition induced by operating of the appropriate finite group. The applied method was introduced by Z. Janko, [4].

2. CONSTRUCTION

Theorem 2.1. *There exist exactly eight nonisomorphic symmetric (100, 45, 20) - designs admitting the operation of the group $G = E_{25} \cdot S_3$ so that E_{25} acts semiregularly, an element of order three has exactly four fixed points and an involution fixes twenty points. The designs are pairwise dual and $E_{25} \cdot S_3$ is their full automorphism group.*

Proof. Let's denote by D a symmetric (100,45,20)-design. We'll prove the first statement together with the explicit construction of designs. The group $G = E_{25} \cdot S_3$ (a semidirect product of the elementary abelian group of order 5^2 and the symmetric group of order 6, $|G| = 150$) is in terms of generators and relations given by

$$G = \langle a, b, c, d \mid a^5 = b^5 = 1, ab = ba, c^3 = 1, d^2 = 1, cd = dc^2, \\ c^{-1}ac = b, c^{-1}bc = a^{-1}b^{-1}, dad = a, dbd = a^{-1}b^{-1} \rangle.$$

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Semiregular acting of $E_{25} = \langle a, b\rangle \leq G$ on D gives that possible G-orbit lengths are 25, 50 and 75. On the orbits of these lengths the number of fixed points for the automorphisms of order 2 and 3 proves to be as follows:

orb. length→	25	50	75
$\langle d\rangle$	5	0	5
$\langle c\rangle$	1	2	0

Now from the conditions on $\langle c\rangle$ and $\langle d\rangle$-acting on D we conclude that on our design G acts semitransitively in four orbits of the length 25 (on points and blocks). The points from orbit $I, I = 1,2,3,4$ we'll denote by $I_1, \ldots, I_{25}$. Thus for our construction we need only a permutation representation of the G-generators of degree 25. The one being applied is given in Table I, section 3.

In the sense of [4] one gets two orbit matrices, each representing a possible tactical decomposition induced by G-acting on D (see [1]):

(2.1)

25	25	25	25	
15	10	10	10	25
10	15	10	10	25
10	10	15	10	25
10	10	10	15	25

and

(2.2)

25	25	25	25	
14	12	11	8	25
12	11	8	14	25
11	8	14	12	25
8	14	12	11	25

To index matrices (2.1) and (2.2) means to specify the points from $\{I_1, \ldots, I_{25}\}$, $I = 1,2,3,4$ which lie on particular D-block, all the blocks included ([4]). Indexing four orbit representative blocks determines our design completely for the other blocks of D can be generated by G-acting on these ones. As an orbit representative block we'll take a block stabilized by subgroup $S_3 = \langle c, d\rangle \leq G$ and therefore consisting of complete $\langle c, d\rangle$-point orbits on $\{I_1, \ldots, I_{25}\}, I = 1, \ldots, 4$. From Table I it turns out that $\langle c, d\rangle$-orbits on 25 points are

$$\{1\}, \{2,6,14\}, \{3,7,19\}, \{4,8,23\}, \{5,9,10\},$$
$$\{11,13,15,16,18,22\} \text{ and } \{12,17,20,21,24,25\}.$$

Thus we readily see that (2.2) is impossible to index.

In indexing (2.1) a significant diminution of the task scope can be obtained by noticing that the full automorphism group of this matrix is isomorphic to S_4 and then implementing the related reductions. Additionally, the isomorphism

α of order 4 defined by relations

$$\alpha^4 = 1, \alpha^{-1} a \alpha = a^2, \alpha^{-1} b \alpha = b^2, \alpha^{-1} c \alpha = c \; and \; \alpha^{-1} d \alpha = d$$

obviously normalizes G, so by its acting on index set isomorphic structures can be reduced. For this purpose we use the permutation representation of degree 25

$$\alpha = (1)(2\ 3\ 5\ 4)(6\ 7\ 9\ 8)(10\ 23\ 14\ 19)(11\ 21\ 22\ 17)(12\ 16\ 20\ 15)(13\ 25\ 18\ 24).$$

The indexing of (2.1) we finally accomplish with the help of a computer and get eight designs. Regarding the statistics of their three and four blocks intersection, they prove to be nonisomorphic. Dualizing the designs we obtain that they are pairwise dual. Hence, under given assumptions, up to isomorphism and duality we've constructed exactly four designs. Them we denote $D_i, i = 1, \ldots, 4$ and give explicitly by the representative blocks l_i^1, l_i^2, l_i^3 and l_i^4 of their four orbits, $i = 1, \ldots, 4$.

$$
\begin{aligned}
&Design D_1\\
l_1^1 = \;& 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{19} 1_{22}\\
& 2_1 2_2 2_3 2_4 2_6 2_7 2_8 2_{14} 2_{19} 2_{23}\\
& 3_1 3_2 3_3 3_5 3_6 3_7 3_9 3_{10} 3_{14} 3_{19}\\
& 4_1 4_2 4_6 4_{11} 4_{13} 4_{14} 4_{15} 4_{16} 4_{18} 4_{22}\\
l_1^2 = \;& 1_1 1_2 1_4 1_5 1_6 1_8 1_9 1_{10} 1_{14} 1_{23}\\
& 2_2 2_3 2_4 2_6 2_7 2_8 2_{12} 2_{14} 2_{17} 2_{19} 2_{20} 2_{21} 2_{23} 2_{24} 2_{25}\\
& 3_1 3_3 3_7 3_{12} 3_{17} 3_{19} 3_{20} 3_{21} 3_{24} 3_{25}\\
& 4_1 4_3 4_4 4_5 4_7 4_8 4_9 4_{10} 4_{19} 4_{23}\\
l_1^3 = \;& 1_1 1_5 1_9 1_{10} 1_{11} 1_{13} 1_{15} 1_{16} 1_{18} 1_{22}\\
& 2_1 2_2 2_4 2_5 2_6 2_8 2_9 2_{10} 2_{14} 2_{23}\\
& 3_2 3_6 3_{11} 3_{12} 3_{13} 3_{14} 3_{15} 3_{16} 3_{17} 3_{18} 3_{20} 3_{21} 3_{22} 3_{24} 3_{25}\\
& 4_1 4_3 4_7 4_{12} 4_{17} 4_{19} 4_{20} 4_{21} 4_{24} 4_{25}\\
l_1^4 = \;& 1_1 1_2 1_3 1_4 1_6 1_7 1_8 1_{14} 1_{19} 1_{23}\\
& 2_1 2_4 2_8 2_{12} 2_{17} 2_{20} 2_{21} 2_{23} 2_{24} 2_{25}\\
& 3_1 3_2 3_6 3_{11} 3_{13} 3_{14} 3_{15} 3_{16} 3_{18} 3_{22}\\
& 4_4 4_8 4_{11} 4_{12} 4_{13} 4_{15} 4_{16} 4_{17} 4_{18} 4_{20} 4_{21} 4_{22} 4_{23} 4_{24} 4_{25}
\end{aligned}
$$

$Design D_2$

$l_2^1 = 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{19} 1_{22}$
$2_1 2_2 2_3 2_4 2_6 2_7 2_8 2_{14} 2_{19} 2_{23}$
$3_1 3_2 3_4 3_5 3_6 3_8 3_9 3_{10} 3_{14} 3_{23}$
$4_1 4_2 4_6 4_{11} 4_{13} 4_{14} 4_{15} 4_{16} 4_{18} 4_{22}$

$l_2^2 = 1_1 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{14} 1_{19}$
$2_2 2_3 2_4 2_6 2_7 2_8 2_{12} 2_{14} 2_{17} 2_{19} 2_{20} 2_{21} 2_{23} 2_{24} 2_{25}$
$3_1 3_3 3_7 3_{12} 3_{17} 3_{19} 3_{20} 3_{21} 3_{24} 3_{25}$ $4_1 4_3 4_4 4_5 4_7 4_8 4_9 4_{10} 4_{19} 4_{23}$

$l_2^3 = 1_1 1_2 1_6 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{22}$
$2_1 2_2 2_4 2_5 2_6 2_8 2_9 2_{10} 2_{14} 2_{23}$
$3_2 3_6 3_{11} 3_{12} 3_{13} 3_{14} 3_{15} 3_{16} 3_{17} 3_{18} 3_{20} 3_{21} 3_{22} 3_{24} 3_{25}$
$4_1 4_3 4_7 4_{12} 4_{17} 4_{19} 4_{20} 4_{21} 4_{24} 4_{25}$

$l_2^4 = 1_1 1_2 1_3 1_4 1_6 1_7 1_8 1_{14} 1_{19} 1_{23}$
$2_1 2_4 2_8 2_{12} 2_{17} 2_{20} 2_{21} 2_{23} 2_{24} 2_{25}$
$3_1 3_5 3_9 3_{10} 3_{11} 3_{13} 3_{15} 3_{16} 3_{18} 3_{22}$
$4_4 4_8 4_{11} 4_{12} 4_{13} 4_{15} 4_{16} 4_{17} 4_{18} 4_{20} 4_{21} 4_{22} 4_{23} 4_{24} 4_{25}$

$Design D_3$

$l_3^1 = 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{19} 1_{22}$
$2_1 2_2 2_3 2_4 2_6 2_7 2_8 2_{14} 2_{19} 2_{23}$
$3_1 3_2 3_4 3_5 3_6 3_8 3_9 3_{10} 3_{14} 3_{23}$
$4_1 4_5 4_9 4_{10} 4_{11} 4_{13} 4_{15} 4_{16} 4_{18} 4_{22}$

$l_3^2 = 1_1 1_2 1_6 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{22}$
$2_4 2_8 2_{11} 2_{12} 2_{13} 2_{15} 2_{16} 2_{17} 2_{18} 2_{20} 2_{21} 2_{22} 2_{23} 2_{24} 2_{25}$
$3_1 3_2 3_3 3_4 3_6 3_7 3_8 3_{14} 3_{19} 3_{23}$
$4_1 4_4 4_8 4_{12} 4_{17} 4_{20} 4_{21} 4_{23} 4_{24} 4_{25}$

$l_3^3 = 1_1 1_3 1_4 1_5 1_7 1_8 1_9 1_{10} 1_{19} 1_{23}$
$2_1 2_3 2_7 2_{12} 2_{17} 2_{19} 2_{20} 2_{21} 2_{24} 2_{25}$
$3_2 3_3 3_4 3_6 3_7 3_8 3_{12} 3_{14} 3_{17} 3_{19} 3_{20} 3_{21} 3_{23} 3_{24} 3_{25}$
$4_1 4_2 4_4 4_5 4_6 4_8 4_9 4_{10} 4_{14} 4_{23}$

$l_3^4 = 1_1 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{14} 1_{19}$
$2_1 2_2 2_6 2_{11} 2_{13} 2_{14} 2_{15} 2_{16} 2_{18} 2_{22}$
$3_1 3_3 3_7 3_{12} 3_{17} 3_{19} 3_{20} 3_{21} 3_{24} 3_{25}$
$4_2 4_6 4_{11} 4_{12} 4_{13} 4_{14} 4_{15} 4_{16} 4_{17} 4_{18} 4_{20} 4_{21} 4_{22} 4_{24} 4_{25}$

DesignD$_4$

$$
\begin{aligned}
l_4^1 = \;& 1_2 1_3 1_5 1_6 1_7 1_9 1_{10} 1_{11} 1_{13} 1_{14} 1_{15} 1_{16} 1_{18} 1_{19} 1_{22}\\
& 2_1 2_2 2_3 2_5 2_6 2_7 2_9 2_{10} 2_{14} 2_{19}\\
& 3_1 3_3 3_4 3_5 3_7 3_8 3_9 3_{10} 3_{19} 3_{23}\ 4_1 4_5 4_9 4_{10} 4_{11} 4_{13} 4_{15} 4_{16} 4_{18} 4_{22}\\
l_4^2 = \;& 1_1 1_5 1_9 1_{10} 1_{11} 1_{13} 1_{15} 1_{16} 1_{18} 1_{22}\\
& 2_2 2_6 2_{11} 2_{12} 2_{13} 2_{14} 2_{15} 2_{16} 2_{17} 2_{18} 2_{20} 2_{21} 2_{22} 2_{24} 2_{25}\\
& 3_1 3_2 3_3 3_5 3_6 3_7 3_9 3_{10} 3_{14} 3_{19}\\
& 4_1 4_4 4_8 4_{12} 4_{17} 4_{20} 4_{21} 4_{23} 4_{24} 4_{25}\\
l_4^3 = \;& 1_1 1_2 1_4 1_5 1_6 1_8 1_9 1_{10} 1_{14} 1_{23}\\
& 2_1 2_3 2_7 2_{12} 2_{17} 2_{19} 2_{20} 2_{21} 2_{24} 2_{25}\\
& 3_3 3_4 3_5 3_7 3_8 3_9 3_{10} 3_{12} 3_{17} 3_{19} 3_{20} 3_{21} 3_{23} 3_{24} 3_{25}\\
& 4_1 4_2 4_3 4_4 4_6 4_7 4_8 4_{14} 4_{19} 4_{23}\\
l_4^4 = \;& 1_1 1_2 1_3 1_4 1_6 1_7 1_8 1_{14} 1_{19} 1_{23}\\
& 2_1 2_2 2_6 2_{11} 2_{13} 2_{14} 2_{15} 2_{16} 2_{18} 2_{22}\\
& 3_1 3_3 3_7 3_{12} 3_{17} 3_{19} 3_{20} 3_{21} 3_{24} 3_{25}\\
& 4_3 4_7 4_{11} 4_{12} 4_{13} 4_{15} 4_{16} 4_{17} 4_{18} 4_{19} 4_{20} 4_{21} 4_{22} 4_{24} 4_{25}
\end{aligned}
$$

The dual of D_i we denote by $D_i^*, i = 1, \dots, 4$.

Let's return to the approval that designs D_i and $D_i^*, i = 1, \dots, 4$ are not isomorphic, which we base on investigation of the intersection of block triplets for each of them and, where necessary, the intersection of four blocks. Seven different statistics counting the number of triplets that intersect in $0, \dots, 20$ points are obtained, those for designs D_1 and D_1^* being identical. Instead of presenting complete statistics, it suffices to give the number of block triplets that intersect in, for instance, 5 points for each design (table (2.3)).

(2.3)

D_1	D_2	D_3	D_4	D_1^*	D_2^*	D_3^*	D_4^*
325	375	1125	750	325	575	625	1000

Additionally, table (2.4) shows that statistics of intersection of four blocks for designs D_1 and D_1^* are different, so one concludes that $D_i, D_i^*, i = 1, \dots, 4$ are nonisomorphic.

(2.4)

Intersec. card.→	0	1	2	3	4	5
D_1	7050	96125	477200	1019250	1228500	756525
D_1^*	8400	96175	461750	1050575	1207975	756575

6	7	8	9	10	11	12	13-20
271875	55650	7750	1250	25	0	25	0
274925	56000	8050	650	125	25	0	0

A computation (V. Tonchev's program) shows that the full automorphism group for all the obtained designs has the order 150. This confirms the validity of the relations $AutD_i \cong AutD_i^* \cong E_{25} \cdot S_3$, $i = 1, \ldots, 4$ and completes the proof of the theorem. □

3. Appendix

TABLE I. The permutation representation of G-generators (degree 25)

generator a --> (no fixed point)
(1 2 3 4 5)(6 10 11 12 13)(7 16 23 25 21)(8 17 24 19 15)
(9 18 20 22 14)

generator b --> (no fixed point)
(1 6 7 8 9)(2 10 16 17 18)(3 11 23 24 20)(4 12 25 19 22)
(5 13 21 15 14)

generator c --> (one fixed point)
(1)(2 6 14)(3 7 19)(4 8 23)(5 9 10)(11 13 15)(12 21 25)
(16 22 18)(17 24 20)

generator d --> (five fixed points)
(1)(2)(3)(4)(5)(6 14)(7 19)(8 23)(9 10)(11 18)(12 20)
(13 22)(15 16)(17 25)(21 24) □

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