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PROPER METRIC SPACES AND HIGSON COMPACTIFICATIONS OF PRODUCT SPACES

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ABSTRACT. Let (X, d) be a non-compact metric space. We provide an equivalent condition that the metric d is proper on X. $\overline{X}^d$ denotes the Higson compactification of a non-compact proper metric space (X, d). In this paper we show that if (X, d_X) is a non-compact proper metric space and (Y, d_Y) is a non-compact proper metric space or a non-degenerate compact metric space, then $\overline{X \times Y}^{\max\{d_X, d_Y\}}$ is not equivalent to $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$.

1. INTRODUCTION AND PRELIMINARIES

The Higson compactification is a compactification which is defined for all locally compact metric spaces endowed with certain metrics [6]. We say that a metric d on X is *proper* provided that every bounded set in X has a compact closure. For X to have a proper metric, obviously X must be locally compact. Let (X, d) be a metric space. In this paper, for $r > 0$, $B_r(x, d)$ and $\text{diam}X$ denote $\{y \in X : d(x, y) < r\}$ and $\sup\{d(x, y) : x, y \in X\}$ respectively. Suppose that X is non-compact with d a proper metric. Let $f : X \to Y$ be a continuous function into a metric space Y with specific metric. We say that a function f satisfies *the* $(*)_d$*-condition* provided that $\lim_{x\to\infty} \text{diam}(f(B_r(x, d))) = 0$ for any $r > 0$. The $(*)_d$-condition means that for each $r > 0$ and each $\varepsilon > 0$, there is a compact set $K = K_{r,\varepsilon}$ in X such that for all $x \notin K$, $\text{diam}(f(B_r(x, d))) < \varepsilon$. We now define $C_d^*(X)$ and $C_d(X)$. Recall that $C(X)$ (resp. $C^*(X)$) denotes the set of all real-valued (resp. bounded real-valued) continuous functions on X. These are rings under pointwise addition and multiplication with $C^*(X)$ a subring of $C(X)$. By analogy with these definitions we define $C_d(X) = \{f \in C(X) : f \text{ satisfies the } (*)_d\text{-condition}\}$ and $C_d^*(X) = \{f \in C^*(X) : f \text{ satisfies the } (*)_d\text{-condition}\}$. With the supremum norm on $C^*(X)$, $C_d^*(X)$ is a closed subring of $C^*(X)$ containing all the constant functions. Because the metric d on X is proper, $C_d^*(X)$ generates the topology of X. It is well-known that the compactifications of X are in one-to-one correspondence with the closed subrings $\mathcal{F}$ of $C^*(X)$ which contain the

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constants and generate the topology of X (cf. [1], Theorem 3.7.). We are now in a position to define the Higson compactification and its corona. The Higson compactification is the compactification associated with the closed subring $\mathcal{F} = C_d^*(X) \subset C^*(X)$ [6]. We denote the Higson compactification by $\overline{X}^d$, which depends on the metric d. The corona of this compactification is the set $\overline{X}^d - X$ with the subspace topology. We denote the corona of X by $\nu_d X$. It is characterized as the compactification $\overline{X}^d$ such that the real-valued continuous functions on X that extend to $\overline{X}^d$ are precisely the ones in $C_d^*(X)$.

The following proposition was shown in [6].

Proposition 1.1. *Supposes that X is non-compact and that d is a proper metric on X. The Higson compactification $\overline{X}^d$ is the unique compactification of X such that if Y is any compact metric space and $f : X \to Y$ is continuous, then f has a continuous extension to $\overline{X}^d$ if and only if f satisfies the $(*)_d$-condition.*

In this paper $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{D}_\kappa$ denote the real line endowed with a usual topology, the set of all positive integers and a discrete space with cardinality κ respectively. In section 2, for non-compact metric space (X, d), we show that the metric d is proper on X if and only if $C_d^*(X)$ separates points from closed subsets of X. Let (X, d_X) and (Y, d_Y) be non-compact proper metric spaces. It is natural to ask a question whether $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$ is equivalent to $\overline{X \times Y}^{\rho}$ for some proper metric ρ compatible with the topology of $X \times Y$. Y. Iwamoto introduced the notion that two proper metrics d and ρ on X are similar (Definition is appeared in section 3.) and he proved that $\overline{X}^d$ is equivalent to $\overline{X}^{\rho}$ if and only if d and ρ are similar. In section 3, we show that if d and $\max\{d_X, d_Y\}$ are similar, then $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$ is not equivalent to $\overline{X \times Y}^d$. Assume that (K, d_K) is a non-degenerate compact metric space. Furthermore, we show that if d and $\max\{d_X, d_K\}$ are similar, then $\overline{X}^{d_X} \times K$ is not equivalent to $\overline{X \times K}^d$. Let (X, d) be a non-compact proper metric space and $n < \omega$. We show that there exists a proper metric ρ on $\mathbb{D}_n \times X$ such that $\overline{\mathbb{D}_n \times X}^{\rho}$ is equivalent to $\mathbb{D}_n \times \overline{X}^d$. In particular, if ρ and ρ' are similar, then $\overline{K \times X}^{\rho'}$ is equivalent to $K \times \overline{X}^d$.

For undefined notation and terminology, see [2] and [3].

2. Proper metric spaces

In this section we assume that (X, d) is a non-compact metric space. A subset $\mathcal{F} \subset C^*(X)$ *separates points from closed subsets of* X provided that for any closed subset $F \subset X$ and any $x \in X - F$ there is an $f \in \mathcal{F}$ with $f(x) \notin \mathrm{cl}_{\mathbb{R}} f(X - F)$. In this section we discuss concerning proper metric spaces. Now, it is easy to see that $C_d^*(X)$ separates points from closed subsets of X if (X, d) is a proper metric space. In this section we show that the converse is true. At first, we will prove the following lemma:

Lemma 2.1. *If $C_d^*(X)$ separates points from closed subsets of X, then*

$$\text{diam}X = +\infty.$$

Proof. Assume the contrary that $\text{diam}X < +\infty$. Then there exists an $s > 0$ such that $B_s(x, d) = X$. In this case, every element of $C_d^*(X)$ is a constant function. In fact, for any $f \in C_d^*(X)$ and any $n \in \mathbb{N}$, there exists a compact subset K_n of X such that $\text{diam}(f(B_s(x, d))) = \text{diam}(f(X)) < 1/n$ if $x \notin K_n$. This indicates that f is a constant function. This contradicts for the fact that $C_d^*(X)$ separates points from closed subsets of X and then the proof is complete. □

Lemma 2.1 shows that if d is a proper metric on X, then $\text{diam}X = +\infty$. From the Lemma 2.1 we will prove the theorem below:

Theorem 2.2. *If $C_d^*(X)$ separates points from closed subsets of X, then d is proper on X.*

Proof. Assume the contrary that d is not proper. Then there exist an $x \in X$ and an $r > 0$ such that $B_r(x, d)$ is not relatively compact. At first, we will show the following claim below:

Claim. For any $y \in B_r(x, d)$, there exists a $\lambda > 0$ such that $B_\lambda(y, d) \supset B_r(x, d)$ and $B_\lambda(y, d) \neq B_r(x, d)$.

In fact, from the Lemma 2.1 $\text{diam}X = +\infty$. Then we can take a point $z \in X - B_r(x, d)$ because $X - B_r(x, d) \neq \emptyset$. Put $\lambda = 2r + d(x, z)$ and then $B_\lambda(y, d) \supset B_r(x, d)$ holds for any $y \in B_r(x, d)$. Furthermore, since $d(y, z) \leq d(y, x) + d(x, z) < r + d(x, z) < \lambda$, $z \in B_\lambda(y, d)$ and thus $B_\lambda(y, d) \neq B_r(x, d)$. Then the proof of claim is complete.

Now, since $B_r(x, d) - K \neq \emptyset$ for any compact subset K of X, we can take a point $y \in B_r(x, d) - K$. Since $C_d^*(X)$ separates points from closed subsets of X, there exists an $f \in C_d^*(X)$ such that $f(x) \notin \text{cl}_{\mathbb{R}} f(X - B_r(x, d))$. Put $\varepsilon_0 = d(f(x), \text{cl}_{\mathbb{R}} f(X - B_r(x, d)))$. Let λ be as in the Claim and $z \in B_\lambda(y, d) - B_r(x, d)$. Since $B_\lambda(y, d) \supset B_r(x, d)$ and $B_\lambda(y, d) \cap (X - B_r(x, d)) \neq \emptyset$, we note that $\text{diam}(f(B_\lambda(y, d))) \geq |f(x) - f(z)| \geq \varepsilon_0$. This is a contradiction and thus d is proper on X. The proof is complete. □

From the Theorem 2.2 we can get the following corollary:

Corollary 2.3. *$C_d^*(X)$ separates points from closed subsets of X if and only if d is proper on X.*

3. Higson compactifications of product spaces

For compactifications αX and γX of X, $\alpha X \geq \gamma X$ if there exists a continuous map $f : \alpha X \to \gamma X$ such that $f \restriction_X$ is an identity on X. αX is equivalent to γX provided that if there is a homeomorphism $f : \alpha X \to \gamma X$ such that $f \restriction_X$ is an identity on X. (We denote this by writing $\alpha X \approx \gamma X$.) Let (X, d) be a metric space and A a subset of X. Let $d(x, A)$ and $B_r(A, d)$ be denoted by $\inf\{d(x, a) : a \in A\}$ and $\{x \in X : d(x, A) < r\}$ respectively. The following definition was introduced by Y. Iwamoto.

Definition 3.1 ([5]). *Let d and ρ be two proper metrics on X. We write $d \prec \rho$ provided that if for any $r > 0$, there exists a compact set K in X and $s_r > 0$ such that $B_r(x, d) \subset B_{s_r}(x, \rho)$ whenever $x \in X - K$. If $d \prec \rho$ and $\rho \prec d$ then we say that d and ρ are similar. (We denote this by writing $d \risingdotseq \rho$.)*

The following lemma was proved by Y. Iwamoto.

Lemma 3.2 ([5]). *Let d and ρ be two proper metrics on non-compact space X. Then $\overline{X}^d \approx \overline{X}^\rho$ if and only if $d \risingdotseq \rho$.*

We will prove the theorem below:

Theorem 3.3. *Let (X, d_X) and (Y, d_Y) be non-compact proper metric spaces. Then $\overline{X \times Y}^d$ is not equivalent to $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$, where $d = \max\{d_X, d_Y\}$.*

Proof. Assume the contrary that $\overline{X \times Y}^d \approx \overline{X}^{d_X} \times \overline{Y}^{d_Y}$. Since both X and Y are non-compact metric spaces, there exist countable infinite closed discrete subsets N and M of X and Y respectively. Let us denote these sets by $N = \{x_n : n < \omega\}$ and $M = \{y_n : n < \omega\}$. We now construct subsets $P \subset N$ and $Q \subset M$ as subsequences. Let $p_0 = x_0$ and $q_0 = y_0$. Since d_X and d_Y on X and Y, respectively, are proper, there must be $x_i \notin B_3(p_0, d_X)$ and $y_i \notin B_3(q_0, d_Y)$. Choose such x_i and y_i and let $p_1 = x_i$ and $q_1 = y_i$. Similarly, there must be $x_i \notin \bigcup_{i<2} B_4(p_i, d_X)$ and $y_i \notin \bigcup_{i<2} B_4(q_i, d_Y)$. Choose such x_i and y_i and let $p_2 = x_i$ and $q_2 = y_i$. Continuing in these fashions we obtain subsets $P = \{p_n : n < \omega\} \subset N$ and $Q = \{q_n : n < \omega\} \subset M$ such that $p_n \notin \bigcup_{k<n} B_{n+2}(p_k, d_X)$ and $q_n \notin \bigcup_{k<n} B_{n+2}(q_k, d_Y)$ for any $n \in \mathbb{N}$. Now, let $D = \{t_n : t_n = (p_n, q_n),\ n < \omega\}$. We will verify that $\mathcal{B} = \{B_{r_n}(t_n, d) : n < \omega\}$ is a discrete open collection of $X \times Y$, where $r_n = (n+1)/2$ for $n < \omega$. In fact, it is sufficient to show that $\{B_{r_n}(p_n, d_X) : n < \omega\}$ and $\{B_{r_n}(q_n, d_Y) : n < \omega\}$ are discrete open collections of X and Y respectively and then we will show the following claim:

Claim 1.

(1) $|\{n : B_{r_n}(p_n, d_X) \cap B_{1/2}(x, d_X) \neq \emptyset \text{ and } n < \omega\}| \leq 1$ holds for any $x \in X$,

(2) $|\{n : B_{r_n}(q_n, d_Y) \cap B_{1/2}(y, d_Y) \neq \emptyset \text{ and } n < \omega\}| \leq 1$ holds for any $y \in Y$.

We will show the Claim 1-(1). In fact, assume the contrary that there exist $i, j < \omega$ and $x \in X$ such that $B_{r_i}(p_i, d_X) \cap B_{1/2}(x, d_X) \neq \emptyset$ and $B_{r_j}(p_j, d_X) \cap B_{1/2}(x, d_X) \neq \emptyset$ hold. Without loss of generality, we may assume that $i < j$.

From the assumption above we can take $z_k \in B_{r_k}(p_k, d_X) \cap B_{1/2}(x, d_X)$ for $k = i,\ j$. Then

$$\begin{aligned} d_X(p_i, p_j) \leq &\ d_X(p_i, z_i) + d_X(z_i, x) + d_X(x, z_j) + d_X(z_j, p_j) \\ \leq &\ \tfrac{i+1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{j+1}{2} < j + 2. \end{aligned}$$

However, this contradicts the choice of p_j, $d_X(p_i, p_j) \geq j + 2$. By the similar argument of Claim 1-(1) we can show the Claim 1-(2).

From the Claim 1 we note the following claim:

Claim 2.

(1) $|\{n : (X \times \{y\}) \cap B_{r_n}(t_n, d) \neq \emptyset \text{ and } n < \omega\}| \leq 1$ holds for any $y \in Y$,
(2) $|\{n : (\{x\} \times Y) \cap B_{r_n}(t_n, d) \neq \emptyset \text{ and } n < \omega\}| \leq 1$ holds for any $x \in X$.

Then we will define a map $f : X \times Y \to [0, 1]$ as follows:

$$f(p) = \begin{cases} 0, & \text{if } p \notin \bigcup_{n<\omega} B_{r_n}(t_n, d), \\ (r_n - d(p, t_n))/r_n, & \text{if } p \in B_{r_n}(t_n, d). \end{cases}$$

Claim 3. $f \in C_d^*(X \times Y)$.

It is sufficient to show that f satisfies the $(*)_d$-condition. In fact, let $\varepsilon > 0$ be fixed and let $r > 0$. Then there exists an $n < \omega$ such that $r_i > 4r/\varepsilon$ for $i \geq n$. Put $K = \bigcup_{i<n} B_{r_i}(t_i, d)$. Since d is proper, $\mathrm{cl}_{X\times Y} B_r(K, d)$ is compact in $X \times Y$.

Now, if $p \notin \mathrm{cl}_{X\times Y} B_r(K, d)$ and $B_r(p, d) \cap B_{r_i}(t_i, d) \neq \emptyset$, then $r_i \geq r_n > 4r/\varepsilon$. We will show that $\mathrm{diam}(f(B_r(p, d))) \leq \varepsilon$. It is sufficient to show that $d(f(p), f(z)) < \varepsilon/2$ for every $z \in B_r(p, d)$. To that end of the proof of Claim 3, it is sufficient to consider the following four cases below:

(1) $z \in B_r(p, d)$, $z \in B_{r_i}(t_i, d)$ and $p \in B_{r_i}(t_i, d)$,
(2) $z \in B_r(p, d)$, $z \in B_{r_i}(t_i, d)$ and $p \notin B_{r_i}(t_i, d)$ for every $i < \omega$,
(3) $z \in B_r(p, d)$ and $p,\ z \notin B_{r_i}(t_i, d)$ for every $i < \omega$,
(4) $z \in B_r(p, d)$, $p \in B_{r_i}(t_i, d)$ and $z \in B_{r_j}(t_j, d)$ for some $i,\ j < \omega$ with $i \neq j$.

In the case (1), $|f(p) - f(z)| = |(r_i - d(p, t_i))/r_i - (r_i - d(z, t_i))/r_i| = |d(z, t_i) - d(p, t_i)|/r_i < r/r_i < \varepsilon/4$. In the case (2), $|f(p) - f(z)| = |f(z)| = |r_i - d(z, t_i)|/r_i < r/r_i < \varepsilon/4$. In the case (3), $|f(p) - f(z)| = 0$. Finally, in the case (4), without loss of generality, we may assume that $i < j$. Then $|f(p) - f(z)| \leq \varepsilon/2$ because $|f(p)| = |r_i - d(p, t_i)|/r_i < r/r_i < \varepsilon/4$ and $|f(z)| = |r_j - d(z, t_j)|/r_j < r/r_i < \varepsilon/4$. This completes the proof that $\mathrm{diam}(f(B_r(p, d))) \leq \varepsilon$. Hence, f satisfies the $(*)_d$-condition and then $f \in C_d^*(X \times Y)$.

From Claim 2 for any $x \in X$, there exists a compact set K_x of Y such that $f(p) = 0$ if $p \in \{x\} \times (Y - K_x)$. Similarly, for any $y \in Y$, there exists a compact set K_y of X such that $f(p) = 0$ if $p \in (X - K_y) \times \{y\}$. Since $\overline{X}^{d_X} \times \overline{Y}^{d_Y} \approx \overline{X \times Y}^{d}$, f has a continuous extension $\overline{f}$ on $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$. R

denotes $\overline{X}^{d_X} \times \overline{Y}^{d_Y} - X \times Y$. From the facts above we note that $\overline{f}(x) = 0$ if $x \in R$. Since $\mathrm{cl}_{\overline{X}^{d_X} \times \overline{Y}^{d_Y}} D \cap R \neq \emptyset$, we take a point $p \in \mathrm{cl}_{\overline{X}^{d_X} \times \overline{Y}^{d_Y}} D \cap R$. Then we can verify that $\sup_{z \in U_p} |\overline{f}(z) - \overline{f}(p)| = 1$ for any neighborhood U_p of p. This contradicts the continuity of $\overline{f}$ and then $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$ is not equivalent to $\overline{X \times Y}^d$. The proof is complete. □

From Lemma 3.2 we can get the following corollary:

Corollary 3.4. *Let (X, d_X) and (Y, d_Y) be non-compact proper metric spaces. If $d = \max\{d_X, d_Y\}$, then $\overline{X \times Y}^d$ is not equivalent to $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$.*

Furthermore, the following theorem holds even if one factor is non- degenerate compact metrizable.

Theorem 3.5. *Let (K, d_K) be a non-degenerate compact metric space, (X, d) a non-compact proper metric space and $\rho = \max\{d, d_K\}$. Then $\overline{X \times K}^\rho$ is not equivalent to $\overline{X}^d \times K$.*

Proof. Assume the contrary that $\overline{X \times K}^\rho \approx \overline{X}^d \times K$. Let r be denoted by a diameter of K and a $k \in K$. We will define a function $f : X \times K \to [0, 1]$ as follows:

$$f((x, z)) = \begin{cases} 0, & \text{if } z \notin B_{s/3}(k, d_K), \\ (s - 3 \cdot d_K(k, z))/s, & \text{if } z \in B_{s/3}(k, d_K), \end{cases}$$

where $s = \min\{r, 1\}$. Then $f \in C^*(X \times K)$. For any $y \in K$ we define $f_y : X \to [0, 1]$ by $f_y(x) = f((x, y))$. Since $f_y \in C_d^*(X)$, there exists a continuous extension $\overline{f_y} : \overline{X}^d \to [0, 1]$ such that $\overline{f_y} \restriction_X = f_y$. We will define a function $\overline{f} : \overline{X}^d \times K \to [0, 1]$ as follows: $\overline{f}((x, y)) = \overline{f_y}(x)$ for any $(x, y) \in \overline{X}^d \times K$. We will verify that $\overline{f} \in C^*(\overline{X}^d \times K)$ with $\overline{f} \restriction_{X \times K} = f$. To do this it is sufficient to show that if $p \in \nu_d X$ and $q \in K$ then $\overline{f} \restriction_{(X \times K) \cup \{(p,q)\}}$ is continuous (cf. 6H of [4]). Let $\varepsilon > 0$ be given. We must find V open in $\overline{X}^d$ and W open in K for which $(p, q) \in V \times W$ and $\overline{f}((V \times W) \cap (X \times K)) \subset (\overline{f}((p, q)) - \varepsilon, \overline{f}((p, q)) + \varepsilon)$. As $\overline{f}_q$ is continuous there exists an open subset V of $\overline{X}^d$ such that $p \in V$ and $\overline{f}_q(V) \subset (\overline{f}_q(p) - \varepsilon/4, \overline{f}_q(p) + \varepsilon/4)$. Thus, if $x \in V$ then

$$\overline{f}((x, q)) \in (\overline{f}((p, q)) - \varepsilon/4, \overline{f}((p, q)) + \varepsilon/4).$$

As f is uniformly continuous there exists $\delta > 0$ such that if (x, v) and (x', w) are in $X \times K$ and $\rho((x, v), (x', w)) < \delta$ then $|f((x, v)) - f((x', w))| < \varepsilon/4$. So, let $W = B_\delta(q, d_K)$. Then if $(x, v) \in (V \cap X) \times W$, then $|f((x, v)) - f((x, q))| < \varepsilon/4$. Combine this with (1) and conclude that $\overline{f}((V \times W) \cap (X \times K)) \subset (\overline{f}((p, q)) - \varepsilon, \overline{f}((p, q)) + \varepsilon)$. Thus $\overline{f}$ is continuous as claimed. Since $\overline{X \times K}^\rho \approx$

$\overline{X}^d \times K$, f satisfies the $(*)_\rho$-condition. On the other hand, we note that $\mathrm{diam}(f(B_{r+1}((y,z),\rho))) = 1$ for any $(y,z) \in X \times K$. This is a contradiction and then the proof is complete. □

From Lemma 3.2 we can get the following corollary:

Corollary 3.6. *Let (K, d_K) be a non-degenerate compact metric space and (X, d) a non-compact proper metric space. If $\rho \doteq \max\{d, d_K\}$, then $\overline{X \times K}^\rho$ is not equivalent to $\overline{X}^d \times K$.*

In Theorem 3.5 if K is discrete, then there exists a proper metric ρ compatible with the topology of K such that $\overline{X \times K}^\rho \approx \overline{X}^d \times K$. The rest of this paper, for the sake of abbreviation, let $\mathbb{D}_n$ be denoted by $\{0, \cdots, n-1\}$ for $n < \omega$.

Theorem 3.7. *Let (X, d) be a non-compact proper metric space. Then for any compact discrete space K there exists a proper metric ρ on $K \times X$ such that $\overline{K \times X}^\rho \approx K \times \overline{X}^d$.*

Proof. Since K is compact discrete, without loss of generality, we can consider that there exists an $n < \omega$ such that $K = \mathbb{D}_n$. Fix an element $x_0 \in X$. Then we will define a metric $\rho : (\mathbb{D}_n \times X) \times (\mathbb{D}_n \times X) \to \mathbb{R}$ as follows:

$$\rho((i,x),(j,y)) = \begin{cases} d(x,y), & \text{if } i = j, \\ \max\{d(x_0,x) + i, d(x_0,y) + j\}, & \text{if } i \neq j. \end{cases}$$

Then it is easy to see that ρ is a proper metric on $\mathbb{D}_n \times X$ and for any $r > 0$, $B_r((i,x),\rho) = \{i\} \times B_r(x,d)$ if $d(x_0,x) \geq r$. Now, let Y be a compact metric space and $f : (\mathbb{D}_n \times X, \rho) \to Y$ a continuous map satisfying the $(*)_\rho$-condition. Define $f_i : X \to Y$ by $f_i(x) = f((i,x))$ and $d_i : X^2 \to \mathbb{R}$ by $d_i(x,y) = \rho((i,x),(i,y))$ for $i < n$. Then we note that $d_i = d$ for $i < n$ and it is easy to see that f_i satisfies the $(*)_d$-condition for $i < n$ and then f_i has a continuous extension $\overline{f_i} : \overline{X}^d \to Y$. Then we will define $\overline{f} : \mathbb{D}_n \times \overline{X}^d \to Y$ as follows:

$$\overline{f}((i,x)) = \overline{f_i}(x), \quad \text{if } x \in \overline{X}^d.$$

Then $\overline{f} : \mathbb{D}_n \times \overline{X}^d \to Y$ is a continuous extension of f.

Conversely, we assume that a continuous map $f : (\mathbb{D}_n \times X, \rho) \to Y$ has a continuous extension $\overline{f} : \mathbb{D}_n \times \overline{X}^d \to Y$. For any $i < n$ let f_i and d_i be defined as in the argument above. For any $i < n$ define $\overline{f_i} : \overline{X}^d \to Y$ by $\overline{f}_i(x) = \overline{f}((i,x))$ for $x \in \overline{X}^d$. Since $\overline{X}^{d_i} \approx \overline{X}^d$, f_i satisfies the $(*)_{d_i}$-condition for $i < n$. To that end of the proof, we will show that f satisfies the $(*)_\rho$-condition. For any $\varepsilon > 0$ be fixed and any $r > 0$, then there exists a compact set K_i of X such that if $x \notin K_i$, then $\operatorname{diam}(f_i(B_r(x,d_i))) < \varepsilon$ for $i < n$. Put $K = \bigcup_{i<n}\{i\} \times (K_i \cup \operatorname{cl}_X B_r(x_0, d_i))$. If $(i,x) \notin K$, then $\operatorname{diam}(f(B_r((i,x),\rho))) = \operatorname{diam}(f(\{i\} \times B_r(x,d_i))) = \operatorname{diam}(f_i(B_r(x,d_i)))$ for $i < n$ and then $\operatorname{diam}(f(B_r((i,x),\rho))) < \varepsilon$. Hence f satisfies the $(*)_\rho$-condition. From the uniqueness of $\overline{\mathbb{D}_n \times X}^{\rho}$, $\overline{\mathbb{D}_n \times X}^{\rho} \approx \mathbb{D}_n \times \overline{X}^{d}$ and then the proof is complete. □

From Lemma 3.2 we can get the following corollary:

Corollary 3.8. *Let (X,d) be a non-compact proper metric space, K a compact discrete space, and ρ a proper metric on $X \times K$ defined on Theorem* 3.7. *If $\rho' \doteqdot \rho$, then $\overline{K \times X}^{\rho'} \approx K \times \overline{X}^{d}$.*

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REFERENCES

[1] B.J. Ball and S. Yokura *Compactifications determined by subsets of $C^*(X)$*, Top. Appl. 13 (1982), 1-13.

[2] R.E. Chandler, *Hausdorff Compactifications*, Marcel Dekker, Inc., New York-Basel, 1976.

[3] R. Engelking, *General Topology*, Helderman Verlag, Berlin, 1989.

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1976.

[5] Y. Iwamoto, *A metric condition which gives the same Higson compactification*, manuscript.

[6] J. Keesling, *The one-dimensional Čech cohomology of the Higson compactification and its corona*, Topology. Proc. 19 (1994), 129-148.

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