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SOME CRYSTAL ROGERS-RAMANUJAN TYPE IDENTITIES

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ABSTRACT. By using the KMN^2 crystal base character formula for the basic $A_2^{(1)}$-module, and the principally specialized Weyl-Kac character formula, we obtain a Rogers-Ramanujan type combinatorial identity for colored partitions. The difference conditions between parts are given by the energy function of certain perfect $A_2^{(1)}$-crystal. We also recall some other identities for this type of colored partitions, but coming from the vertex operator constructions and with no apparent connection to the crystal base theory.

1. INTRODUCTION

J. Lepowsky and R. Wilson gave in [LW] a Lie-theoretic interpretation of Rogers-Ramanujan identities in terms of representations of affine Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})^{\sim}$. The product sides of Rogers-Ramanujan identities follow from the principally specialized Weyl-Kac character formula for level 3 standard $\tilde{\mathfrak{g}}$-modules, the sum sides follow from the vertex operator construction of bases of level 3 standard $\tilde{\mathfrak{g}}$-modules, parameterized by partitions satisfying difference 2 conditions.

The Lepowsky-Wilson approach is also possible for other affine Lie algebras and for other constructions of vertex operators, various combinatorial consequences are illustrated by constructions given, for example, in [C3], [LP], [Ma] and [Mi].

In [P1] there is a construction of the basic $A_\ell^{(1)}$-module based on the Frenkel-Kac vertex operator formula. It was noted there that the combinatorial difference conditions arising from the vertex operator formula coincide

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with the energy function in the construction by paths of the basic representation of $A_{\ell-1}^{(1)}$ given in [DJKMO]. In [P2] this combinatorial connection for basic modules is extended to other classical affine Lie algebras, this time by using more general crystal base character formula for standard modules due to S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki [KMN^2].

This combinatorial connection of the crystal base theory and the vertex operator constructions suggested that it might be interesting to study Rogers-Ramanujan type combinatorial identities for colored partitions, where differences are given by energy functions of perfect crystals. So we start, roughly speaking, with colored partitions $(n_1)_{\beta_1} \geqslant (n_2)_{\beta_2} \geqslant \cdots \geqslant (n_s)_{\beta_s} > 0$, where each number n_r is "colored" with a "color" β_r from the set of nine "colors" $\{1, \ldots, 9\}$. Analogous to the Rogers-Ramanujan case, we consider colored partitions satisfying difference conditions

$$(n_r)_{\beta_r} \geqslant (n_{r+1})_{\beta_{r+1}} + E_{\beta_r \beta_{r+1}} ,$$

where differences $E_{\beta_r\beta_{r+1}} \in \{0,1,2\}$ are the values of an energy function of certain perfect $\mathfrak{sl}(3,\mathbb{C})^{\sim}$-crystal. We obtain an identity for such colored partitions (Theorem 2.1) by using the principally specialized Weyl-Kac character formula and the crystal base character formula [KMN^2].

In the last section we recall some other identities for this type of colored partitions, but coming from the vertex operator construction [MP2] and with no apparent connection to the crystal base theory.

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2. A COMBINATORIAL IDENTITY

Let A be a nonempty set and denote by $\mathcal{P}(A)$ the set of all maps $\pi : A \to \mathbb{N}$, where $\pi(a)$ equals zero for all but finitely many $a \in A$. Clearly π is determined by its values $(\pi(a) \mid a \in A)$ and we shall also write π as a monomial

$$\pi = \prod_{a \in A} a^{\pi(a)}.$$

We shall say that π is a partition and for $\pi(a) > 0$ we shall say that a is a part of π. We define the length $\ell(\pi)$ of π by $\ell(\pi) = \sum_{a\in A} \pi(a)$. We consider elements of A as partitions of length 1, i.e. $A \subset \mathcal{P}(A)$. For $\rho, \pi \in \mathcal{P}(A)$ we write $\pi \supset \rho$ if $\pi(a) \geqslant \rho(a)$ for all $a \in A$ and we say that π contains ρ.

For $\rho, \pi \in \mathcal{P}(A)$ we define $\pi\rho$ in $\mathcal{P}(A)$ by $(\pi\rho)(a) = \pi(a) + \rho(a)$, $a \in A$. We shall say that $1 = \prod_{a\in A} a^0$ is the partition with no parts and length 0.

Clearly $\mathcal{P}(A)$ is a monoid. For lack of a better terminology, we shall say that $\mathcal{I} \subset \mathcal{P}(A)$ is an ideal in the monoid $\mathcal{P}(A)$ if $\rho \in \mathcal{I}$ and $\pi \in \mathcal{P}(A)$ implies $\rho\pi \in \mathcal{I}$. For such an $\mathcal{I}$ we call the difference of sets $\mathcal{P}(A)\backslash\mathcal{I}$ a partition ideal in $\mathcal{P}(A)$ (cf. [A1, Chapter 8]). Later on we shall consider an ideal $\mathcal{I} = \mathcal{P}(A)\mathcal{D}$ generated by a set $\mathcal{D}$, and the corresponding partition ideal we shall denote as

$$\mathcal{P}_{\mathcal{D}} = \mathcal{P}(A)\backslash(\mathcal{P}(A)\mathcal{D}).$$

Let $a \mapsto |a|$ be a map from A to $\mathbb{N}$. Then we define the degree $|\pi|$ of π by

$$|\pi| = \sum_{a\in A} |a|\,\pi(a),$$

and we say that a part a of π has the degree $|a|$.

Now let

$$\Gamma = \{1,2,3,4,5,6,7,8,9\},$$

we shall think of Γ as a set of colors, and let $A = \Gamma \times \mathbb{Z}_{<0}$. We shall write $(\alpha, i) = i_\alpha$,

$$A = \{i_\alpha \mid \alpha \in \Gamma, i \in \mathbb{Z}_{<0}\},$$

and we shall think that i_α has a color α. We define a map $(-i)_\alpha \mapsto |(-i)_\alpha|$ for $i > 0$ by

$$\begin{aligned} &(-i)_1 \mapsto 3i-2, \qquad && (-i)_4 \mapsto 3i, \qquad && (-i)_9 \mapsto 3i+2,\\ &(-i)_2 \mapsto 3i-1, && (-i)_5 \mapsto 3i, && (-i)_8 \mapsto 3i+1,\\ &(-i)_3 \mapsto 3i-1, && (-i)_6 \mapsto 3i, && (-i)_7 \mapsto 3i+1. \end{aligned} \tag{2.1}$$

So, for example, $(-5)_1$ has the color 1 and the degree $|(-5)_1| = 13$. In general, we can think of $\pi \in \mathcal{P}(A)$ as a colored partition of the nonnegative integer $n = |\pi|$.

Let $E = (E_{\alpha\beta})_{\alpha,\beta\in\Gamma}$ be a matrix

$$E = \begin{pmatrix} 2&2&2&1&2&2&2&2&2\\ 1&2&1&1&2&1&2&2&2\\ 1&1&2&1&1&2&2&2&2\\ 1&1&1&0&1&1&1&1&1\\ 0&0&1&1&0&1&1&2&2\\ 0&1&0&1&1&0&2&1&2\\ 0&1&0&1&1&0&2&1&2\\ 0&0&1&1&0&1&1&2&2\\ 0&0&0&1&0&0&1&1&2 \end{pmatrix}.$$

Define a subset $\mathcal{D}$ in $\mathcal{P}(A)$ by

$$\mathcal{D} = \{i_\alpha i_\beta \mid E_{\alpha\beta}E_{\beta\alpha} \geqslant 1\} \bigcup \{(i-1)_\alpha i_\beta \mid E_{\alpha\beta} = 2\}.$$

We shall say that a colored partition π satisfies the difference $\mathcal{D}$ condition if π does not contain any colored partition from $\mathcal{D}$, or, equivalently, if π is an element of the partition ideal $\mathcal{P}_{\mathcal{D}} = \mathcal{P}(A)\backslash(\mathcal{P}(A)\mathcal{D})$. So, for example, $\pi = (-5)_1(-3)_8(-2)_9$ does not satisfy the difference $\mathcal{D}$ condition since $E_{89} = 2$ and $(-3)_8(-2)_9 \in \mathcal{D}$.

Now we can state the following Rogers-Ramanujan type identity:

Theorem 2.1.

$$\sum_{n=0}^{\infty} \sharp\{|\pi| = n \mid \pi \in \mathcal{P}_{\mathcal{D}}\} \, q^n = \prod_{r=1}^{\infty} (1-q^r)^{-1}.$$

The product side follows from the principally specialized Weyl-Kac character formula for the basic $\mathfrak{sl}(3,\mathbb{C})^{\sim}$-module [L], the sum side follows from the crystal base character formula [KMN^2]. The proof is given in the next section.

3. The principally specialized character for the basic $A_2^{(1)}$-module

Let us consider a multiple of the character of the basic $\mathfrak{sl}(3,\mathbb{C})^{\sim}$-module $L(\Lambda_0)$

$$\prod_{r=1}^{\infty} (1-e^{-r\delta})^{-1} \cdot e^{-\Lambda_0} \operatorname{ch} L(\Lambda_0). \tag{3.1}$$

Then the principal specialization $e^{-\alpha_i} \mapsto q$, $i = 0, 1, 2$, of this product gives

$$\prod_{r=1}^{\infty} (1-q^{3r})^{-1} \prod_{r \not\equiv 0 \bmod 3} (1-q^r)^{-1} = \prod_{r=1}^{\infty} (1-q^r)^{-1}.$$

Here we use notions, notation and results as in [K] or [L].

On the other side, we may use the results in [KMN^2] to express (3.1) in terms of colored partitions. We consider a perfect crystal Γ for $\mathfrak{sl}(3,\mathbb{C})^{\sim}$ coming from the tensor product of the vector representation and its dual:

$$\begin{tikzcd}
1 \arrow[r, "1"] \arrow[d, "2"'] & 2 \arrow[r, "2"] & 5 \arrow[d, "2"'] \\
3 \arrow[r, "1"] & 6 \arrow[d, "1"'] & 8 \arrow[d, "1"'] \\
4 & 7 \arrow[r, "2"] & 9
\end{tikzcd}$$

together with 0-arrows

$$9 \xrightarrow{0} 4 \xrightarrow{0} 1\,, \quad 8 \xrightarrow{0} 3 \quad \text{and} \quad 7 \xrightarrow{0} 2\,.$$

For $\beta \in \Gamma$ let us denote by $\mathrm{wt}(\beta)$ the $\mathfrak{h}$-weight of β, that is, the restriction of the classical weight of β on the fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Note that

$$\begin{aligned} \mathrm{wt}\,1 &= -\mathrm{wt}\,9 = \alpha_1 + \alpha_2, \\ \mathrm{wt}\,2 &= -\mathrm{wt}\,8 = \alpha_2, \\ \mathrm{wt}\,3 &= -\mathrm{wt}\,7 = \alpha_1, \\ \mathrm{wt}\,4 &= \mathrm{wt}\,5 = \mathrm{wt}\,6 = 0. \end{aligned}$$

The crystal Γ has the unique energy function H with values in $\{0, 1, 2\}$, and we have chosen

$$E_{\alpha\beta} = H(\beta \otimes \alpha).$$

In particular, we have

$$E_{44} = H(4 \otimes 4) = 0, \qquad \mathrm{wt}\,4 = 0. \tag{3.2}$$

The ground state path for the basic module $L(\Lambda_0)$ is

$$p_{\Lambda_0} = 4, 4, 4, 4, 4, 4, \ldots .$$

By [KMN^2, Proposition 4.6.4], the set $\mathcal{P}(\Lambda_0, \Gamma)$ of sequences $p = \big(p(j)\,;\, j \geqslant 1\big)$ in Γ such that $p(j) = p_{\Lambda_0}(j) = 4$ for $j \gg 0$ parameterizes a basis of $L(\Lambda_0)$, where, by taking into account (3.2), the weight and the degree of a sequence is given by

$$|p| = -\sum_{j=1}^{\infty} j\, H\big(p(j+1) \otimes p(j)\big), \tag{3.3}$$

$$\mathrm{wt}(p) = \sum_{j=1}^{\infty} \mathrm{wt}\big(p(j)\big). \tag{3.4}$$

We want to interpret this result in terms of colored partitions, and, in order to do that, let us think of colored partitions in a different way: Let $\preccurlyeq$ be an order on Γ defined as

$$1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6 \succ 7 \succ 8 \succ 9,$$

and define an order on A by

$$i_\beta \preccurlyeq j_\gamma \qquad \text{if} \qquad \text{either} \quad i < j \quad \text{or} \quad i = j,\, \beta \preccurlyeq \gamma.$$

This is a total order on A. For a colored partition

$$\nu = (j_1)_{\beta_1} \cdots (j_s)_{\beta_s} \in \mathcal{P}(A)$$

we may assume that $j_1 \leqslant \cdots \leqslant j_s < 0$ and that $j_r = j_{r+1}$ implies $\beta_r \preccurlyeq \beta_{r+1}$, i.e., $(j_r)_{\beta_r} \preccurlyeq (j_{r+1})_{\beta_{r+1}}$. Sometimes we shall denote a colored partition ν as

$$\nu = \big((j_1)_{\beta_1} \preccurlyeq (j_2)_{\beta_2} \preccurlyeq \cdots \preccurlyeq (j_s)_{\beta_s}\big).$$

We may visualize ν by its Young diagram:

$$\begin{array}{llll} \text{color} & \beta_1 & \square\square\cdots\square\cdots\square\square & -j_1 \quad \text{boxes} \\ & \beta_2 & \square\square\cdots\square\cdots\square & -j_2 \\ & & \cdots & \\ & \beta_s & \square\square\cdots\square & -j_s\,. \end{array}$$

Then the total number of boxes $||\nu||$ in the Young diagram of ν is

$$||\nu|| = \sum_{m=1}^{s} -j_m\,.$$

Since to each color $\beta \in \Gamma$ we can associate its weight $\mathrm{wt}(\beta) \in \mathfrak{h}^*$, we define a weight $\mathrm{wt}(\nu)$ of the colored partition ν as

$$\mathrm{wt}(\nu) = \sum_{m=1}^{s} \mathrm{wt}(\beta_m)\,.$$

Now for a given path

$$p = \beta_1, \ldots, \beta_{s-1}, \beta_s, \ldots, 4, 4, 4, 4, \ldots$$

we construct a colored partition $\mathrm{part}_{\mathcal{D}}(p)$ in the following way: We start with a large enough s, i.e., an s such that $\beta_j = 4$ for all $j \geqslant s$, and we set

$$-i_s = -i_{s+1} = \cdots = 0,$$

and from there on

$$\begin{aligned} &-i_{s-1} = E_{\beta_{s-1}\beta_s}, \\ &-i_{s-2} = E_{\beta_{s-2}\beta_{s-1}} + E_{\beta_{s-1}\beta_s}, \\ &\cdots \\ &-i_1 = \sum_{r=1}^{s-1} E_{\beta_r\beta_{r+1}}. \end{aligned}$$

Note that $E_{44} = 0$, so it does not matter with which $s \gg 0$ we have started. Now we define a colored partition $\mathrm{part}_{\mathcal{D}}(p)$ associated to p as

$$\mathrm{part}_{\mathcal{D}}(p) = (-i_1)_{\beta_1} \ldots (-i_{s-1})_{\beta_{s-1}} (-i_s)_{\beta_s} \ldots (0)_4 (0)_4 (0)_4 (0)_4 \ldots,$$

where we identify the product of all $(0)_4$ with $1 \in \mathcal{P}(A)$. It is easy to check that

$$E_{\beta_r\beta_{r+1}} = 0 \quad \text{implies} \quad \beta_r \preccurlyeq \beta_{r+1}, \tag{3.5}$$

so by construction we have

$$\mathrm{part}_{\mathcal{D}}(p) = \big((-i_1)_{\beta_1} \preccurlyeq (-i_2)_{\beta_2} \preccurlyeq \dots\big).$$

We may visualize the above construction of $p \mapsto \mathrm{part}_{\mathcal{D}}(p)$ in terms of Young diagrams: We start from "the bottom" for large enough s and we add 0 boxes to a color $\beta_s = 4$. As we are finished with associating $-i_{r+1}$ boxes to β_{r+1}, we associate to β_r an extra $E_{\beta_r,\beta_{r+1}}$ boxes, so that $\mathrm{part}_{\mathcal{D}}(p)$ satisfies difference conditions

$$-i_r = -i_{r+1} + E_{\beta_r,\beta_{r+1}}.$$

By counting the number of boxes we added at each stage, we see that the total number of boxes in $\mathrm{part}_{\mathcal{D}}(p)$ equals $-|p|$ given by (3.3), i.e.

$$\|\mathrm{part}_{\mathcal{D}}(p)\| = \sum_{r=1}^{s} r\, E_{\beta_r,\beta_{r+1}} = -|p|. \tag{3.6}$$

We also have (cf. (3.4) and (3.2))

$$\mathrm{wt}(\mathrm{part}_{\mathcal{D}}(p)) = \sum_{r=1}^{s} \beta_r = \mathrm{wt}(p). \tag{3.7}$$

For a given colored partition ν and a plain partition $\Delta \in \mathcal{P}(\mathbb{Z}_{<0})$ (i.e. a partition "without colors"),

$$\begin{aligned}\nu &= \big((i_1)_{\beta_1} \preccurlyeq \dots \preccurlyeq (i_s)_{\beta_s}\big),\\ \Delta &= (j_1 \leqslant j_2 \leqslant \dots \leqslant j_p),\end{aligned}$$

$i_s, j_p < 0$, we define a colored partition $\nu \oplus \Delta$ by

$$\nu \oplus \Delta = (k_1)_{\beta_1} \dots (k_r)_{\beta_r},$$

where $r = \max\{s,p\}$, with additional colors $\beta_{s+1} = \dots = \beta_p = 4$ in the case $p > s$, and

$$k_n = i_n + \sum_{-j_m \geqslant n} -1,$$

that is, to a Young diagram of ν we add to each color $\beta_1, \dots, \beta_{-j_1}$ one box, then we add to each color $\beta_1, \dots, \beta_{-j_2}$ another one box, and so on. Of course, in the case $p > s$ we consider $i_{s+1} = \cdots = i_p = 0$. It is clear that in the case $\nu = \operatorname{part}_{\mathcal{D}}(p)$ we have difference conditions

$$-i_r \geqslant -i_{r+1} + E_{\beta_r, \beta_{r+1}}.$$

It is clear that the map

$$(p, \Delta) \mapsto \operatorname{part}_{\mathcal{D}}(p) \oplus \Delta, \qquad \mathcal{P}(\Lambda_0, \Gamma) \times \mathcal{P}(\mathbb{Z}_{<0}) \to \mathcal{P}(A)$$

is injective. Hence, by using (3.6) and (3.7), [KMN[2], Proposition 4.6.4] implies

$$\sum_{\pi = \operatorname{part}_{\mathcal{D}}(p) \oplus \Delta} e^{\operatorname{wt}(\pi) - \|\pi\|\delta} = \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-1} \cdot e^{-\Lambda_0} \operatorname{ch} L(\Lambda_0),$$

where the sum runs over all pats $p \in \mathcal{P}(\Lambda_0, \Gamma)$ and plain partitions $\Delta \in \mathcal{P}(\mathbb{Z}_{<0})$.

What we want to see is that every $\pi = \operatorname{part}_{\mathcal{D}}(p) \oplus \Delta$ is in $\mathcal{P}_{\mathcal{D}}$. For that it is sufficient to show that

(3.8) $$(i_\alpha \preccurlyeq j_\beta) \in \mathcal{P}_{\mathcal{D}} \quad \text{if and only if} \quad |i - j| \geqslant E_{\alpha\beta},$$

(3.9)
$$i_\alpha \preccurlyeq j_\beta \preccurlyeq k_\gamma, \ |i - k| \leqslant 1, \ i_\alpha k_\gamma \in \mathcal{D} \quad \text{implies} \quad i_\alpha j_\beta \in \mathcal{D} \quad \text{or} \quad j_\beta k_\gamma \in \mathcal{D}.$$

It is easy to see, by using (3.5), that (3.8) holds. Now, in the presence of (3.8), we can easily check that (3.9) holds as well; in terms of E it reads as

(3.10) $$E_{\alpha\gamma} \leqslant E_{\alpha\beta} + E_{\beta\gamma},$$

or, equivalently, as

$$E_{\alpha\gamma} \leqslant E_{\alpha\beta} \quad \text{if} \quad E_{\beta\gamma} = 0, \qquad E_{\alpha\gamma} \leqslant E_{\beta\gamma} \quad \text{if} \quad E_{\alpha\beta} = 0.$$

Hence every $\pi = \operatorname{part}_{\mathcal{D}}(p) \oplus \Delta$ is in $\mathcal{P}_{\mathcal{D}}$. Moreover, every $\pi \in \mathcal{P}_{\mathcal{D}}$ can be written in this way, and hence we have

$$\sum_{\pi \in \mathcal{P}_{\mathcal{D}}} e^{\operatorname{wt}(\pi) - \|\pi\|\delta} = \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-1} \cdot e^{-\Lambda_0} \operatorname{ch} L(\Lambda_0).$$

Since the map $(-i)_\alpha \mapsto |(-i)_\alpha|$ defined by (2.1) is the principal specialization, Theorem 2.1 holds.

4. Some remarks

Let Γ be a classical crystal with an energy function H with values in the set $\{0,1,2\}$. Then, as before, we can consider colored partitions $\pi \in \mathcal{P}(A)$ with $A = \Gamma \times \mathbb{Z}_{<0}$ and we can define the degree $||\pi||$ and the $\mathfrak{h}$-weight of π as above, with $\mathfrak{h} = \mathbb{C}\text{-span}\{h_1, \ldots, h_\ell\}$ (see [KMN²]). If we set $E_{\alpha\beta} = H(\beta \otimes \alpha) \in \{0,1,2\}$, we can define

$$\mathcal{D} = \{i_\alpha i_\beta \mid E_{\alpha\beta}E_{\beta\alpha} \geqslant 1\} \bigcup \{(i-1)_\alpha i_\beta \mid E_{\alpha\beta} = 2\},$$

and we can consider colored partitions which satisfy difference $\mathcal{D}$ conditions, that is, $\pi \in \mathcal{P}_\mathcal{D}$. Let us define a "character" of the partition ideal $\mathcal{P}_\mathcal{D}$ as

$$\operatorname{ch}(\mathcal{P}_\mathcal{D}) = \sum_{\pi \in \mathcal{P}_\mathcal{D}} e^{\operatorname{wt}(\pi) - ||\pi||\delta}.$$

The map $e^{-\alpha_i} \mapsto q$, $i = 0, 1, \ldots, \ell$, defines $e^{-\delta} \mapsto q^m$ and the principally specialized character

$$\operatorname{ch}_q(\mathcal{P}_\mathcal{D}) = \sum_{n=0}^{\infty} \sharp\{|\pi| = n \mid \pi \in \mathcal{P}_\mathcal{D}\}\, q^n,$$

where $|(-j)_\beta| = mj - \langle \rho, \beta \rangle$, with $\langle \rho, \alpha_i \rangle = 1$ for $i = 1, \ldots, \ell$ (cf. [K]).

With this notation at hand we can write Theorem 2.1, for our particular choice of the $A_2^{(1)}$-crystal Γ, as

$$\operatorname{ch}_q(\mathcal{P}_\mathcal{D}) = \prod_{r=1}^{\infty} (1 - q^r)^{-1}.$$

Let Γ be a perfect crystal for $\tilde{\mathfrak{g}} = \mathfrak{sl}(n, \mathbb{C})^{\sim}$, $n \geq 4$, coming from the tensor product of the vector representation and its dual. Then there is an energy function H taking values in $\{0,1,2\}$ and the ground state path for the basic $\tilde{\mathfrak{g}}$-module is a constant sequence $p(j) = a$, with $E_{aa} = H(a \otimes a) = 0$ and $\operatorname{wt}(a) = 0$. Moreover, there is a total order $\preccurlyeq$ on Γ such that (3.5) holds. So the same proof would go through, and the above identity would hold, if we could show (3.10). For example, in the case of $A_3^{(1)}$-crystal

$$
\begin{tikzcd}
14 \arrow[r, "1"] \arrow[d, "3"'] & 24 \arrow[r, "2"] \arrow[d, "3"'] & 34 \arrow[r, "3"] & 44 \arrow[d, "3"'] \\
13 \arrow[r, "1"] \arrow[d, "2"'] & 23 \arrow[r, "2"] & 33 \arrow[d, "2"'] & 43 \arrow[d, "2"'] \\
12 \arrow[r, "1"] & 22 \arrow[d, "1"'] & 32 \arrow[r, "3"] \arrow[d, "1"'] & 42 \arrow[d, "1"'] \\
11 & 21 \arrow[r, "2"] & 31 \arrow[r, "3"] & 41
\end{tikzcd}
$$

together with 0-arrows

$$41 \xrightarrow{0} 11 \xrightarrow{0} 14\,, \quad 21 \xrightarrow{0} 24\,, \quad 31 \xrightarrow{0} 34\,, \quad 42 \xrightarrow{0} 12\,, \quad 43 \xrightarrow{0} 13$$

all these properties hold, including (3.10), and we have an identity of the above form. On the other side, let $\Gamma = \{1, 2, 3, 4\}$ be the $A_1^{(1)}$-crystal

$$
\begin{tikzcd}
1 \arrow[r, "1"] & 3 \arrow[d, "1"'] \\
2 \arrow[u, "0"'] & 4 \arrow[l, "0"]
\end{tikzcd}
$$

with the energy matrix

$$E = \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

The KMN^2 crystal base character formula is proved under the assumption that the rank of $\mathfrak{g}$ is at least two, but still many results also hold for $\tilde{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C})^{\sim}$. So it is reasonable to ask whether an analogue of Theorem 2.1 holds as well, i.e., whether

$$\mathrm{ch}_q(\mathcal{P}_{\mathcal{D}}) = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \quad ?$$

Note that here the principal specialization reads

$$(-i)_1 \mapsto 2i - 1, \quad (-i)_2 \mapsto 2i, \quad (-i)_3 \mapsto 2i, \quad (-i)_4 \mapsto 2i + 1.$$

What is surprising is that there are other identities of a similar form, but which are not related to the crystal base theory, at least not in any obvious way: consider an "almost perfect" $A_1^{(1)}$-crystal

$$1 \underset{0}{\overset{1}{\rightleftarrows}} 2 \underset{0}{\overset{1}{\rightleftarrows}} 3$$

with the energy matrix

$$E = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} .$$

Then we have

$$\mathrm{ch}_q(\mathcal{P}_\mathcal{D}) = \prod_{r \text{ odd}} (1-q^r)^{-1}. \tag{4.1}$$

Note that here the principal specialization reads

$$(-i)_1 \mapsto 2i-1, \quad (-i)_2 \mapsto 2i, \quad (-i)_3 \mapsto 2i+1.$$

Moreover, if we define a map $(-i)_\alpha \mapsto |(-i)_\alpha|$ for $i > 0$ by

$$(-i)_1 \mapsto 3i-2, \quad (-i)_2 \mapsto 3i, \quad (-i)_3 \mapsto 3i+2,$$

(i.e. if we take the (1,2)-specialization of $\mathrm{ch}(\mathcal{P}_\mathcal{D})$), then we have a Capparelli identity (see [C1]–[C3], [A2])

$$\sum_{n=0}^{\infty} \sharp\{|\pi| = n \mid \pi \in \mathcal{P}_\mathcal{D}\}\, q^n = \prod_{r\equiv 1,3,5,6 \mod 6} (1+q^r). \tag{4.2}$$

Both (4.1) and (4.2) are proved in [MP2] as specializations of the character formula for the basic $\mathfrak{sl}(2,\mathbb{C})^\sim$-module written in the form

$$e^{-\Lambda_0}\,\mathrm{ch}\, L(\Lambda_0) = \mathrm{ch}(\mathcal{P}_\mathcal{D}).$$

The character formula itself is proved by using the Lepowsky-Wilson approach, the proof being quite parallel to [MP1]. The set $\mathcal{D}$ is originally defined as the set of leading terms for the vertex operator algebra defining relations for the basic $\mathfrak{sl}(2,\mathbb{C})^\sim$-module (cf. [MP2, Section 6]).

So it seems that interesting combinatorial properties of $\mathrm{ch}(\mathcal{P}_\mathcal{D})$ go beyond the crystal base character formula [KMN[2], Proposition 4.6.4], at least when the relation $E_{\alpha\gamma} \leqslant E_{\alpha\beta} + E_{\beta\gamma}$ holds and when there is a total order $\preccurlyeq$ on Γ

such that $E_{\alpha\beta} = 0$ implies $\alpha \preccurlyeq \beta$. Further indications for this provide the results in [P2] and the examples below.

Although the one by one matrices have nothing to do with crystals, the notion of difference $\mathcal{D}$ condition still makes sense for $\Gamma = \{1\}$ and $E = (E_{11}) = (2)$; it is simply the difference 2 condition and $\mathrm{ch}_q(\mathcal{P}_\mathcal{D})$, defined via $|(-i)_1| = i$, is the sum side of a Rogers-Ramanujan identity. On the other hand, for $E = (E_{11}) = (1)$ the difference $\mathcal{D}$ condition defines partitions in distinct parts. Of course, the second case is much simpler, and it has an equally simple analogue for $\Gamma = \{1, 2\}$ and $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: consider an "almost perfect" $A_1^{(1)}$-crystal

$$1 \underset{0}{\overset{1}{\rightleftarrows}} 2$$

with the energy matrix $E_{\alpha\beta} = H(\beta \otimes \alpha)$ chosen to be E. Then

$$(-j)_1 \mapsto 2j - \tfrac{1}{2}, \qquad (-j)_2 \mapsto 2j + \tfrac{1}{2}$$

is the principal specialization and we have an identity for partitions in half-integers

$$\mathrm{ch}_q(\mathcal{P}_\mathcal{D}) = \prod_{r \geqslant 1} (1 + q^{r + \frac{1}{2}}).$$

In the case $E = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$ the principal specialization of $\mathcal{P}_\mathcal{D}$ gives partitions in half-integers satisfying difference 3 condition, but I am not aware of any formula that would express $\mathrm{ch}_q(\mathcal{P}_\mathcal{D})$ as an infinite product.

Finally, the formulation of Rogers-Ramanujan type identities for colored partitions in terms of energy functions for crystals was also motivated by a desire to understand better an identity for the basic $A_2^{(1)}$-module obtained in [MP3]. The set $\mathcal{D}$ of difference conditions is defined as the set of leading terms of relations for the basic module, but can be defined as well in the following way: consider the weighted A_2-crystal $\Gamma' = \Gamma \backslash \{4\}$

$$\begin{tikzcd}
1 \arrow[r, "1"] \arrow[d, "2"'] & 2 \arrow[r, "2"] & 5 \arrow[d, "2"'] \\
3 \arrow[r, "1"] & 6 \arrow[d, "1"'] & 8 \arrow[d, "1"'] \\
 & 7 \arrow[r, "2"] & 9
\end{tikzcd}.$$

It is not possible do define 0-arrows which would turn Γ' into an $A_2^{(1)}$-crystal with an energy function. But still, we can define a "difference condition matrix" $(E_{\alpha\beta})_{\alpha,\beta\in\Gamma'}$

$$E = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

and we can define a subset $\mathcal{D}$ in $\mathcal{P}(A)$, $A = \Gamma' \times \mathbb{Z}_{<0}$, by

$$\begin{aligned}\mathcal{D} =& \{i_\alpha i_\beta \mid E_{\alpha\beta}E_{\beta\alpha} \geqslant 1\} \bigcup \{(i-1)_\alpha i_\beta \mid E_{\alpha\beta} = 2\} \\ & \bigcup \{(i-1)_3\, i_5\, i_1\} \bigcup \{(i-1)_9\, (i-1)_5\, i_7\}.\end{aligned}$$

Then, as proved in [MP3], we have

$$e^{-\Lambda_0}\, \mathrm{ch}\ L(\Lambda_0) = \mathrm{ch}(\mathcal{P}_\mathcal{D}).$$

By taking the principal specialization (2.1) without the color 4, we obtain a combinatorial identity

$$\mathrm{ch}_q(\mathcal{P}_\mathcal{D}) = \prod_{r \not\equiv 0 \bmod 3} (1-q^r)^{-1}.$$

As it happens, the above "difference 2 conditions" $\{(i-1)_\alpha i_\beta \mid E_{\alpha\beta} = 2\}$ are the same as in the case of partitions discussed in Section 2. One is tempted to think that this is more than a mere coincidence and that the difference conditions $\{(i-1)_3\, i_5\, i_1\} \bigcup \{(i-1)_9\, (i-1)_5\, i_7\}$ are some sort of "corrections" of the fact that Γ' is not a perfect $A_2^{(1)}$-crystal. With this regard it may be interesting to note that the energy matrix E in Section 2 is invariant under the change

$$1 \leftrightarrow 1, \quad 2 \leftrightarrow 3, \quad 5 \leftrightarrow 6, \quad 7 \leftrightarrow 8, \quad 9 \leftrightarrow 9$$

and $4 \leftrightarrow 4$. Here we have $E_{51} = 0$, $E_{61} = 1$ and $E_{95} = 0$, $E_{96} = 1$. So the "interaction" between parts $i_6\, i_1 \in \mathcal{D}$ and $i_5\, i_1 \notin \mathcal{D}$ is not symmetrical, the later is "compensated" with a weaker requirement $(i-1)_3\, i_5\, i_1 \in \mathcal{D}$. Likewise $i_9\, i_6 \in \mathcal{D}$ "corresponds" to a weaker $i_9\, i_5\, (i+1)_7 \in \mathcal{D}$. It should be said that the present proof of the above identity has no connections with the crystal base theory.

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