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ON THE POWER PROPERTY OF THE DENSITY TOPOLOGY IN THE PLANE

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ABSTRACT. We prove that the density topology in the plane does not have the power property.

1. Introduction.

Any continuous transformation on a topological space X will be called a *mapping*. We say that $f : A \to X$ is *light* on $A \subset X$ if f is non-constant on any non-degenerate continuum in A.

We say that a topology on X has the *power property* if for any open set $U \subset X$, any open and light mapping $f : U \to X$ and any point $x \in U$ there exist a number $n \in \mathbb{N}$ and a neighborhood $V \subset U$ of x such that for each $y \in f(V) \setminus f(\{x\}))$ the set $f^{-1}(\{y\})$ has cardinality n. In other words the power property of a topology means that each open and light mapping on any open set is locally (punctured neighborhoods) n to one .

We recall the *power property* of the Euclidean topology in the plane originally due to Stoïlow. (see [3], [4],[5, Chap. VII, 5.1, p. 88]):

Theorem 1.1. *Let A and B be 2−manifolds and $f(A) = B$ be light and open. For any ordinary point $q \in B$ and any $p \in f^{-1}(q)$, there exists a closed 2−cell neighborhood E of p and an integer k such that $f \restriction E$ is topologically equivalent to $w = z^k$ on $|z| \leq 1$.*

This implies a simple observation on the holomorphic functions (see [5, Chap. VII, Theorem 5.3, p. 88]):

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Corollary 1.2. *The mapping generated by a non-constant differentiable function $w = f(z)$ in a region $R \subset \mathbb{C}$ is locally equivalent to a power mapping.*

A similar result closely related to the power property was obtained by B. Fuglede in [2, Proposition 4.3, p. 292] for the case of the fine topology in potential theory. He proved that a finely holomorphic function is finely locally equivalent to a power mapping. It would be of interest to know whether one can obtain a power property for even finer topology than for the fine topology from potential theory.

In connection with this remark let us formulate two problems.

Problem 1.3. *Which topological spaces have the power property ?*

Problem 1.4. *Does the fine topology from potential theory have the power property?*

We show that the density topology on $\mathbb{C}$ (the density being measured using the Lebesgue measure λ and the discs centered at a point) does not have the power property.

2. Density topology example.

We start with a simple observation

Family lemma 2.1. *For any $t \in (0,1)$ we define the 'family ' $H_t = \{t/2^k : k \geq 0\}$. Given the 'prison' set $M \subset (0,1)$ of the Lebesgue measure $\lambda(M) = 1/2^n$ there exists $t \in (0,1)$ with a 'free family' $F_t = H_t \setminus M$ of cardinality at least n.*

Proof. The assertion is obviously fulfilled with $M = (0, 1/2^n)$. We can try to avoid the free family of cardinality at least $n+1$ with a prison of measure $1/2^n$. For each $t \in (1/2, 1)$ there must be at most n members of the family $H_t = \{t/2^k : k \geq 0\}$ free (i.e. outside the prison M). The most efficient way is to build the prison $M = (0, 1/2^n)$. □

Proposition 2.2. *The density topology in the plane does not have the power property.*

Proof. We set $D = \{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, 0 < t < 2\}$ and define a 'corkscrew'-type mapping $f : D \to \mathbb{C}$ by the formula

$$f(r(\cos \pi t + i \sin \pi t)) = r(\cos 2\pi\varphi(t) + i \sin 2\pi\varphi(t)) \quad ,$$

where $\varphi(t) = t - 1$ for $t \in (1,2]$, $\varphi(t) = \varphi(2^{n+1}t)$ for $t \in (1/2^{n+1}, 1/2^n]$, $n \geq 0$. (With t decreasing from 2 to 1 the 'lower half' of D is mapped onto D counter-clockwise, then the 'speed' of rotation increases in such a way that

$$f(\{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1/2^{n+1}, 1/2^n)\}) = D$$

for $n \geq 0$.)

We consider the mapping $f : D \to \mathbb{C}$ with the density topology on both D and $\mathbb{C}$. We see that

(i) D is a density open set (the missing segment has the Lebesgue measure zero);
(ii) f is density continuous at $D \setminus \{0\}$ (f is piecewisely a rotation in $D \setminus \{0\}$);
(iii) f is density continuous at 0;

Proof of (iii) : For any density open set V containing 0, the density of a set $f^{-1}(V)$ at 0 can be calculated using the 'radial segments'

$$\{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1/2^{n+1}, 1/2^n]\}$$

of D defined by the segments $t \in (1/2^{n+1}, 1/2^n]$, the density of V at 0 gives the density of $f^{-1}(V)$ at 0. ♡

(iv) f is density open at $D \setminus \{0\}$ (D is piecewisely a rotation in $D \setminus \{0\}$);
(v) f is density open at $\{0\}$;

Proof of (v) : The density of U at 0 gives the estimate of the Lebesgue measure of

$$U \cap \{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < 1, t \in (1, 2)\}$$

and we obtain the estimate of the density of $f(U)$ at 0. ♡

Moreover, f is light on D. Hence we summarize :

(vi) f is a light and open mapping on an open set D on a topological space $\mathbb{C}$ with the density topology;

Finally we prove that

(vii) for any density open set $V \subset D$ containing 0 and given $n \in \mathbb{N}$ there exists $y \in f(V)$ such that the set $V \cap f^{-1}(\{y\})$ has cardinality at least n.

Proof of (vii) : There is a density open set $U \subset V$ containing 0 and a Euclidean open set G containing the density closed set $\mathbb{C} \setminus V$ such that G and U are disjoint - see [1], the Lusin-Menchoff property of the density topology. When U reaches the density $1 - 1/2^n$ at 0 for some $R \in (0, 1)$, i.e.

$$\lambda(\{r(\cos \pi t + i \sin \pi t) \in \mathbb{C} : 0 \leq r < R, t \in (0, 2)\}) > (1 - 1/2^n)\pi R^2,$$

we can using the polar coordinates obtain $r \in (0, R)$ such that the set

$$M = \{t \in (0, 2) : r(\cos \pi t + i \sin \pi t) \in G\}$$

is Euclidean open in $(0,2)$ (since G is Euclidean open in $\mathbb{C}$) with $\lambda(M) \le 1/2^n$. Using Family lemma 2.1 with M as the prison set we conclude that there exists $t \in (0,1)$ with the 'free family' set $F_t = \{t_1, \cdots, t_n\} \subset (0,1)$ disjoint with M, being of cardinality at least n. Then $y = f(r(\cos \pi t_1 + i \sin \pi t_1)) = \cdots = f(r(\cos \pi t_n + i \sin \pi t_n)) \in f(V)$ due to the definitions of F_t and f, and consequently $f^{-1}(\{y\})$ has cardinality at least n. ♡

The mapping $f : D \to \mathbb{C}$ shows that the density topology does not have the power property. □

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