GLASNIK MATEMATIČKI
Vol. 34(54)(1999),91–96

THE SUN TOPOLOGY IN THE PLANE

PAVEL PYRIH

ABSTRACT.
We study properties of the *sun topology* which one is somewhere between the *fine topology* from potential theory in the plane and the *density topology* in the plane.

1. Introduction.

The three well known topologies in the plane, the Euclidean topology, the fine topology from potential theory and the usual density topology are linearly ordered. The fine topology is strictly finer than the Euclidean one, the density topology is strictly finer than the density one. The fine topology has several useful geometrical properties which disappear when we switch to the finer density topology.

We introduce the *sun topology* which is somewhere between the *fine topology* from potential theory in the plane and the *density topology* in the plane (the density at any point x is measured using the Lebesgue measure λ and discs $D(x,r)$ with radius r centered at x).

We introduce the *sun topology* in the plane

Definition 1.1. A (Lebesgue) measurable set A in the complex plane is said to be *sun open* if for each $x \in A$ for (Lebesgue) almost every $\alpha \in [0,2\pi]$ there exists $t_\alpha > 0$ such that $\{z \in \mathbb{C} : z = x + t(\cos\alpha + i\sin\alpha), |t| < t_\alpha\} \subset A$.

In other words the set A contains with any point x a segment on almost every line through x.

We show that the sun topology is finer than the fine topology and coarser than the density topology. The sun topology is not normal but keeps the some useful properties of the fine topology.

* Research supported by the grant GAUK 186/96 of Charles University.
1991 *Mathematics Subject Classification.* 54G20,54C10,54A10.
Key words and phrases. Fine topology, density topology.

For any topology (e.g. blue) we use the terms blue open, blue closure ... with respect to this topology.

2. Sun topology.

The following proposition relates the sun topology to the density topology in the plane.

Proposition 2.1. *The density topology is strictly finer than the sun topology.*

Proof. (i) Let the set A be sun open at x. For each $r > 0$ denote by B_r the set of all $\beta \in [0, 2\pi]$ for which $\{z \in \mathbb{C} : z = x + t(\cos\beta + i\sin\beta), |t| < r\} \subset A$ holds. We see that $\lambda_*(B_r) \to 2\pi$ for $r \to 0$. Hence

$$\frac{\lambda_*(A \cap D(x,r))}{\lambda D(x,r)} \geq \frac{\lambda_*(B_r)}{2\pi} \to 1$$

for $r \to 0$. Hence x is a point of density of A. The set A is density open at x. The density topology is finer than the sun topology.

(ii) The set $\mathbb{C} \setminus \{z \in \mathbb{C} : 1/|z| \in \mathbb{N}\}$ is density open at 0 and is not sun open at 0. The density topology is strictly finer than the sun topology. □

Now we introduce the *fine topology* from potential theory in the plane. If we denote for an arbitrary point $x \in \mathbb{C}$, a set $A \subset \mathbb{C}$ and $n \in \mathbb{N}$

$$A_n(x) = \{z \in \mathbb{C} \setminus A, \frac{1}{2^n} \leq |z - x| < \frac{2}{2^n}\} \quad ,$$

we can characterize points x at which a set A is finely open as those for which the series

$$\sum_{n=1}^{\infty} \frac{-n}{\log cap^* (A_n(x))}$$

converges (*Wiener's test*), where *cap** denotes the *outer logarithmic capacity* (see [1, Th.IX,10]).

We will recall ([6,Theorem 4.4, p. 16]) a useful property of the fine topology in the plane

Proposition 2.2. *Every point z of a fine domain U has a fine neighborhood V in U such that any two points a,b of V can be joined by a polygonal path γ in U (not necessarily in V) consisting of just two straight segments of equal length, the length of γ being less than $\alpha|b-a|$, $\alpha > 1$ being given.*

In fact there is a positive probability for a Brownian particle starting at a point x of a fine domain U to reach into any prescribed fine neighborhood of another point y of U before leaving U. (See [7].)

The finely open set U contains with any point x segments starting at x in almost any direction due to

Proposition 2.3. *The sun topology is strictly finer than the fine topology.*

Proof. (i) Let $0 \in A$, A be finely open at 0. We now restrict to $M = \{z = x + iy \in \mathbb{C} : x \geq 0, y \geq 0\}$ and we show that for (Lebesgue) almost every $\alpha \in [0, \pi/2]$ there exists $t_\alpha > 0$ such that $\{z \in \mathbb{M} : z = t(\cos\alpha + i\sin\alpha), 0 \leq t < t_\alpha\} \subset A$. Then with the same information from the other quadrants we conclude that A is sun open at 0. For $n \in \mathbb{N}$ denote

$$A_n = \{z \in \mathbb{M} \setminus A, \frac{1}{2^n} \leq |z| < \frac{2}{2^n}\} \quad .$$

Let $B_n = P(A_n)$, where P is the projection from the origin on the line through the points $[\frac{1}{2^n}, 0]$ and $[0, \frac{1}{2^n}]$. Similarly, let $C_n = R(A_n)$, where R is the projection from the origin on the circle $\{z \in \mathbb{C} : |z| = \frac{1}{2^n}\}$.

The outer logarithmic capacity decreases under contractions, hence

$$cap^*(B_n) \leq cap^*(A_n).$$

For the measurable linear set B_n we can use the estimate the (linear) Lebesgue measure on the line through the points $[\frac{1}{2^n}, 0]$ and $[0, \frac{1}{2^n}]$ by $\lambda(B_n) \leq 4cap^*(B_n)$ (see [4, p. 173]). There is a simple Lipschitz mapping from B_n onto C_n, hence the (linear) Lebesgue measure of C_n on the circle $\{z \in \mathbb{C} : |z| = \frac{1}{2^n}\}$ is estimated by $\lambda(C_n) \leq 2\lambda(B_n)$.

The set A is finely open at 0, hence the series

$$\sum_{n=1}^{\infty} \frac{-n}{log\, cap^*(A_n)}$$

converges. We conclude that the series

$$\sum_{n=1}^{\infty} \frac{-n}{log\, \lambda(C_n)}$$

converges. We know that $\lambda(C_n) \in [0, \frac{\pi}{2 \cdot 2^n}]$. Each point at C_n 'stops' the radius from origin at the distance $\frac{1}{2^n}$. The measure of the angles, which stop at C_n, is $q_n = 2^n \lambda(C_n)$ (belongs to $[0, \pi/2]$). We show that

$$\sum_{n=1}^{\infty} q_n$$

converges. (Suppose not. If $q_n \geq 2^{-n}$ for infinitely many n, then $\lambda(C_n) \geq 2^{-2n}$ for infinitely many n and the series

$$\sum_{n=1}^{\infty} \frac{-n}{log\, \lambda(C_n)}$$

diverges - a contradiction).

We see that the measure of angles in $[0, \pi/2]$ which stop at some distance due to C_n (and due to A_n) for $n \geq m$ is smaller then $\sum_{n=m}^{\infty} q_n$ - hence arbitrarily small for m big enough. Hence (Lebesgue) almost every angle at $[0, \pi/2]$ stops at some positive distance. The set A is sun open at 0.

(ii) The set $\mathbb{C} \setminus \mathbb{R}_+$ is obviously sun open at the origin and is not finely open at the origin due to the fact that each finely open set contains with any point arbitrarily small circles (not discs) centered at the point (see [2, Th.10.14]). The sun topology is strictly finer than the fine topology. □

3. Properties of the sun topology.

We prove some useful properties of the sun topology

Proposition 3.1. *The sun topology has the following properties:*

(i) the sun topology fulfills the essential radius condition, this means that for each $x \in \mathbb{C}$ and each sun neighborhood U of x there is an "essential radius" $r(x, U) > 0$ such that

$$|x - y| \leq min(r(x, U_x), r(y, U_y)) \Rightarrow U_x \cap U_y \neq 0$$

for every sun neighborhoods U_x, U_y of x and y in $\mathbb{C}$,

(ii) the sun topology has the Euclidean G_δ - insertion property , this means that for each sun open set $\mathcal{G}$ and each sun closed set $\mathcal{F}$ with $\mathcal{G} \subset \mathcal{F}$, there is a set G of type Euclidean G_δ such that $\mathcal{G} \subset G \subset \mathcal{F}$,

(iii) any countable set is sun isolated,

(iv) the sun topology is not separable,

(v) the sun topology is locally connected.

Proof. (i) Let U be a sun neighborhood of x. There exist $\alpha_n \in \mathbb{R}$ and $t_n > 0$, $0 \leq n \leq 3$ such that $|\alpha_n - n\pi/2| < 1/11$ and $\{z \in \mathbb{C} : z = x + t(\cos\alpha + i \sin\alpha), |t| < t_n\} \subset U$. Then the essential radius $r(x, U) = min(t_1, t_2, t_3, t_4)/2$. The essential radius condition obviously holds.

(ii) Due to (i) (see [5, 2.D.16, p. 66]).

(iii) Let $\{x_n\}_{n=1}^{\infty}$ be a countable sun dense set. For $x \in \mathbb{C}$ there is a fine neighborhood U of x disjoint with $\{x_n\}_{n=1}^{\infty}$ (any countable set is finely closed and the fine topology is regular - see [5]). The set U is sun open due to Proposition 2.3. Hence the set $\{x_n\}_{n=1}^{\infty}$ is not sun dense.

(iv) See (iii).

(v) The polygonal connectedness of any sun neighborhood follows from the definition of the sun topology. □

Let us remark that the sun topology has some properties similar to the *crosswise topology* - see [5, Ex. 1.1, p. 7].

We recall from [8, Theorem 2.2, p. 347], the following

Proposition 3.2. *Let the topology blue be finer than the topology green on X. Let the topology blue have the green G_δ - insertion property. Suppose $\mathcal{A}$ and $\mathcal{B}$ are disjoint blue closed sets, $\mathcal{U}$ and $\mathcal{V}$ disjoint blue open sets, $\mathcal{A} \subset \mathcal{U}$, and $\mathcal{B} \subset \mathcal{V}$. Then there exist A and B of type green F_σ such that $\mathcal{A} \subset A$, $\mathcal{B} \subset B$, A is disjoint with $\mathcal{B}$, and B is disjoint with $\mathcal{A}$.*

Now we can prove

Proposition 3.3. *The sun topology is not normal.*

Proof. Assume that the sun topology is normal. Let C be the Cantor (middle thirds) set in $[0,1]$. Then for each $a \subset C$, the sets a and $b = C \setminus a$ are (disjoint) sun closed sets (C is a Lebesgue null set).

The sun topology has the Euclidean G_δ - insertion property due to Proposition 3.1 (ii). Proposition 3.2 imply that there exists F of type Euclidean F_σ such that $a \subset F$, F disjoint with b.

The mapping $a \mapsto F$ is an injective mapping (we see that $F \cap C = a$) from the potential set $\mathcal{P}(C)$ to the collection of all Euclidean Borel sets, hence the cardinality argument applies. The cardinality of the collection of all Euclidean Borel sets is c whereas the cardinality of $\mathcal{P}(C)$ equals 2^c - a contradiction. The sun topology is not normal. □

Remark 3.4. The sun topology under the name the *core-a.e.* topology was already introduced by G. Horbaczewska (cf. [3, p. 416]). Moreover, it has quite recently been proved by E. Wagner-Bojakowska and W. Wilczyński that the sun topology is not regular (see [9, Theorem 2, p. 363]).

References

1. M. Brelot, *On Topologies and Boundaries in Potential Theory*, Lecture Notes in Mathematics No. 175, Springer-Verlag, Berlin, 1971.
2. L. L. Helms, *Introduction to Potential Theory*, Wiley Interscience Pure and Applied Mathematics 22, New-York, 1969.
3. G. Horbaczewska, *Core density topologies*, Real Anal. Exchange **20(2)** (1994/95), 416 - 417.
4. N. S. Landkof, *Foundation of Modern Potential Theory*, Springer-Verlag, Berlin, 1972.
5. J. Lukeš, J. Malý, L. Zajíček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Mathematics 1189, Springer-Verlag, Berlin, 1986.
6. T. Lyons, *Finely holomorphic functions*, J. Funct. Anal. **37** (1980), 1 - 18.
7. Nguyen - Xuan - Loc, T. Watanabe, *A characterization of fine domains for a certain class of Markov processes with applications to Brelot harmonic spaces*, Z. Wahrscheinlichkeitshteorie u. verw. Geb. **21** (1972), 167 - 178.
8. P. Pyrih, *Separation of finely closed sets by finely open sets*, Real Anal. Exchange **21(1)** (1995/96), 345 – 348.
9. E. Wagner-Bojakowska, W. Wilczyński, *Separation axioms for some modifications of the core topology*, Atti. Sem. Fis. Univ. Modena **XLVI** (1998), 361 – 370.

(Received: 9.3.1998.)

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague , Czech Republic
E-mail address: pyrih@karlin.mff.cuni.cz