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SOME NEW SYMMETRIC DESIGNS FOR (256,120,56)

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ABSTRACT. We prove the existence of two symmetric designs with parameters (256,120,56) for which the full automorphism group is isomorphic to $(Z_{17}.Z_8) \times (Z_7.Z_3)$.

1. INTRODUCTION

It is our main intention in this paper to prove the following

THEOREM 1.1. *Let G_1 be the semidirect product of cyclic groups of order* 17 *and* 8. *Let G be the direct product of G_1 with the cyclic group of order* 7. *Then there are precisely two mutually nonisomorphic selfdual symmetric designs with parameters* $(256, 120, 56)$ *on which the group G acts in such a way that its cyclic subgroups of order* 17, 8 *and* 7 *keep* 1, 16 *and* 18 *points and blocks fixed respectively.*

We assume that the reader is familiar with the basic notions of design theory, which can be found for example in [3] or [6]. Shortly, a *symmetric design of Menon type* is a finite incidence structure consisting of the set of *points* $\mathcal{P}$, set of *blocks* (*lines*) $\mathcal{B}$ and an incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$ with the following properties: *(i)* $|\mathcal{P}| = |\mathcal{B}| = 4u^2$, *(ii)* Every block is incident with $2u^2 - u$ points, and *(iii)* Every pair of points is incident with $u^2 - u$ blocks. We denote such an incidence structure by $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ and write the belonging parameters as a triple $(4u^2, 2u^2 - u, u^2 - u)$. Between all these designs special importance is given to those of them for which is $u = 2^m$, $m \in \mathbb{N}$ (see [2],[5]). For $m = 2$, many designs have been constructed already (see e.g. [7]). Here we are in the case of $m = 3$, where no other results, except for the whole Menon series, have been known yet.

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We shall use the method of constructing symmetric designs assuming that a collineation group acts on them. It is well known that the orbit partition of points and blocks of a symmetric design under an action of an automorphism group forms a tactical decomposition for this design. More precisely, let $G \leq Aut\mathcal{D}$ be an automorphism group of a symmetric design $\mathcal{D}$ with parameters $(256, 120, 56)$. By [6, Theorem 3.3] G has the same number of orbits on the set of points $\mathcal{P}$ and on the set of blocks $\mathcal{B}$. Denote this number by t. Further, denote the point orbits by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$ and the block orbits by $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_t$. Put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Clearly, $\sum_{r=1}^{t} \omega_r = \sum_{i=1}^{t} \Omega_i = 256$. Let γ_{ir} be the number of points of $\mathcal{P}_r$ which are incident with every block of $\mathcal{B}_i$. These integers γ_{ir} form a tactical decomposition for the orbits $\mathcal{P}_r$ and $\mathcal{B}_i$, $1 \leq i, r \leq t$. Thus, the following equations must hold:

$$\sum_{r=1}^{t} \gamma_{ir} = 120 \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir} \gamma_{jr} = 56 \cdot \Omega_j + \delta_{ij} \cdot 64 \tag{2}$$

DEFINITION 1.2. *The $(t \times t)$-matrix (γ_{ir}), the rows of which satisfy equations* (1) – (2), *is called the orbit structure of the design $\mathcal{D}$.*

So, the first step in the construction of $\mathcal{D}$ admitting the action of G is to determine all possible orbit lengths and to find the orbit structures related to that orbit lengths. In the second step, often called *indexing*, we have to specify which γ_{ir} points of the point orbit $\mathcal{P}_r$ lie on the lines of the block orbit $\mathcal{B}_i$, taking into account that each two blocks have to intersect in 56 points. We define an *index set* as a γ_{ir}-element subset of points of $\mathcal{P}_r = \{r_0, r_1, \ldots, r_{\omega_r - 1}\}$ for every i. Obviously, we can denote these elements with numbers $0, 1, \ldots, \omega_r - 1$, $\forall r$. It is enough to determine a representative for each block orbit; the other blocks of that orbit can be obtained by producing all G-images of that representative. If possible, we choose the block representatives in such a manner that they are stabilized by a subgroup of G different from $\langle 1 \rangle$. For a more detailed explanation of this method of construction, the reader is referred to [4].

2. THE ACTION OF $(Z_{17}.Z_8) \times Z_7$

Denote by $\mathcal{D}$ a (256,120,56) design. For an automorphism φ of $\mathcal{D}$ we denote by $F(\varphi)$ the number of points and blocks fixed by φ.

PROPOSITION 2.1. *Let $\rho \in Aut\mathcal{D}$ with $o(\rho) = 17$. Then $F(\rho) = 1$.*

PROOF. Using the well known upper bound for the number of fixed points of an automorphism, $F(\varphi) \leq k + \sqrt{k - \lambda}$, and the congruence of $F(\rho)$ with

256 mod $o(\rho)$, we get $F(\rho) \in \{1, 18, 35, \ldots, 119\}$. It is an easy combinatorial task to eliminate all the possibilities except for $F(\rho) = 1$. □

LEMMA 2.2. *Let* $G_1 \cong \langle \rho, \sigma \mid \rho^{17} = 1, \sigma^8 = 1, \rho^\sigma = \rho^2 \rangle$ *be the nonabelian group of order* 136. *Let* G_1 *act on* $\mathcal{D}$ *in such a way that* $F(\sigma) = 16$. *Then there exist a unique orbit structure which describes the given action of* G_1.

PROOF. Because of the preceding Proposition, there is not more than one orbit of G_1 on $\mathcal{D}$ of length 1. Further, all other orbit lengths of G_1 must be multiples of 17, since $1 < \Omega_i < 17$ leads to $F(\rho) = \Omega_i$, which is a contradiction. Hence, using the assumption $F(\sigma) = 16$, we get that the only possible partition of orbit lengths is $\{1, 17, 17, \ldots, 17\}$.

If we compute the permutation representation of the generators ρ and σ on 17 elements, we get

$$\begin{aligned} (3) \qquad \rho &= (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16) \\ (4) \qquad \sigma &= (0)(1,2,4,8,16,15,13,9)(3,6,12,7,14,11,5,10) \end{aligned}$$

Using the obvious fact that the stabilizer of each orbit of lenght 17 is a group isomorphic to $\langle \sigma \rangle$, we get an additional assumption for the entrances γ_{ir} of the orbit matrix

$$\gamma_{ir} \equiv 0, 1 \pmod 8 .$$

This assumption together with the system (1)-(2) gives a unique solution for the orbit structure:

$$\begin{array}{ccccccccccc}
1 & 17 & 17 & \cdots & 17 & 17 & 0 & 0 & \cdots & 0 & 0 \\
1 & 9 & 9 & \cdots & 9 & 1 & 8 & 8 & \cdots & 8 & 8 \\
1 & 9 & 9 & \cdots & 1 & 9 & 8 & 8 & \cdots & 8 & 8 \\
\vdots & \vdots & & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & 9 & 1 & \cdots & 9 & 9 & 8 & 8 & \cdots & 8 & 8 \\
1 & 1 & 9 & \cdots & 9 & 9 & 8 & 8 & \cdots & 8 & 8 \\
0 & 8 & 8 & \cdots & 8 & 8 & 9 & 9 & \cdots & 9 & 1 \\
0 & 8 & 8 & \cdots & 8 & 8 & 9 & 9 & \cdots & 1 & 9 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & \vdots \\
0 & 8 & 8 & \cdots & 8 & 8 & 9 & 1 & \cdots & 9 & 9 \\
0 & 8 & 8 & \cdots & 8 & 8 & 1 & 9 & \cdots & 9 & 9
\end{array}$$

Herewith the lemma is proved. □

LEMMA 2.3. *Let* $G \cong G_1 \times \langle \tau \rangle$, $o(\tau) = 7$. *Let* G *act on* $\mathcal{D}$ *with respect to the action of* G_1 *on* $\mathcal{D}$ *given in Lemma* 2.2. *Then there is a unique orbit structure which describes the action of* G *on* $\mathcal{D}$.

PROOF. Possible orbit lengths of G are 1, 17 and 119. As τ fixes all the points of an orbit of lenght 17, and $F(\tau) < 128$, it follows immediately that there are one orbit of length 1, two orbits of length 119 and one orbit of length 17 (the point and block orbits can be ordered in this way without loosing generality). From the orbit structure for G_1 we get immediately the orbit structure for G:

$$\begin{array}{cccc} 1 & 119 & 0 & 0 \\ 1 & 55 & 56 & 8 \\ 0 & 56 & 55 & 9 \\ 0 & 56 & 63 & 1 \end{array}$$

We see that τ fixes 18 points and blocks. □

3. COMPUTER RESULTS AND PROOF OF THEOREM 1.1

Our first intention was to find all (256,120,56) designs acted upon G_1 as described in Lemma 2.2. For all nontrivial orbits (of length 17), (4) yields that there are 5 possibilities for the index sets:

$$\bar{0} = \{0\},\ \bar{1} = \{1,2,4,8,9,13,15,16\},\ \bar{2} = \{3,5,6,7,10,11,12,14\},$$
$$\bar{3} = \{0,1,2,4,8,9,13,15,16\},\ \bar{4} = \{0,3,5,6,7,10,11,12,14\}.$$

Obviously, there are 2^{210} block orbit representatives to be checked on pairwise λ-intersection. Although using strong means for the elimination of isomorphic structures (lexicographical ordering, automorphism group of the orbit structure, outer automorphism $\alpha : x \mapsto 3x \,(\mathrm{mod}\, 17)$), thousands of designs came out as results of a computer program. Hence, we were compelled to add the additional assumption that an automorphism τ of order 7 acts on the design $\mathcal{D}$ by joining seven orbits of length 17 in orbits of length 119 twice. In other words, we assume that $\mathcal{D}$ is acted upon G as described in Lemma 2.3.

Still, it is easier to start the indexing procedure with the orbit structure for G_1. Then, the necessary condition for the existence of an additional automorphism of order 7 is that the multisets of index sets of the first and second 7 block representatives are equal. This fact helped us to decrease the number of possibilities and of course the number of designs. After a few hours computer time we got only two designs which show to be nonisomorphic. We list them here in the form of 15×15 matrices of index sets (we have put away the first row and column of the orbit structure which are trivially indexed with full orbits):

Symmetric design $\mathcal{D}_1$

$$
\begin{array}{ccccccccccccccc}
\bar{3} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{4} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{3} & \bar{4} & \bar{4} & \bar{3} & \bar{4} & \bar{0} & \bar{3} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\
\bar{4} & \bar{3} & \bar{4} & \bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{2} \\
\bar{4} & \bar{4} & \bar{3} & \bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} \\
\bar{3} & \bar{4} & \bar{0} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\
\bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{2} \\
\bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} \\
\bar{2} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{3} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{0} \\
\bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{3} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{0} & \bar{3} \\
\bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{1} & \bar{3} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{0} & \bar{3} & \bar{4} \\
\bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{3} & \bar{4} & \bar{4} & \bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{3} \\
\bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{3} & \bar{0} & \bar{3} & \bar{3} & \bar{4} & \bar{4} \\
\bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{3} & \bar{3} & \bar{0} & \bar{4} & \bar{3} & \bar{4} & \bar{4} & \bar{3} \\
\bar{1} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{3} & \bar{0} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{3} & \bar{4} \\
\bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{0} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4}
\end{array}
$$

Symmetric design $\mathcal{D}_2$

$$
\begin{array}{ccccccccccccccc}
\bar{3} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{4} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{3} & \bar{4} & \bar{4} & \bar{3} & \bar{4} & \bar{0} & \bar{3} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\
\bar{4} & \bar{3} & \bar{4} & \bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{2} \\
\bar{4} & \bar{4} & \bar{3} & \bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{2} \\
\bar{3} & \bar{4} & \bar{0} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} \\
\bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\
\bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} \\
\bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{4} & \bar{0} \\
\bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{3} & \bar{4} & \bar{4} & \bar{3} & \bar{3} & \bar{4} & \bar{0} & \bar{3} \\
\bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{3} & \bar{3} & \bar{4} & \bar{4} & \bar{4} & \bar{0} & \bar{3} & \bar{3} \\
\bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{1} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{0} & \bar{3} & \bar{4} & \bar{3} \\
\bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{3} & \bar{3} & \bar{4} & \bar{0} & \bar{3} & \bar{3} & \bar{4} & \bar{4} \\
\bar{1} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{3} & \bar{4} \\
\bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{3} & \bar{0} & \bar{3} & \bar{4} & \bar{3} & \bar{4} & \bar{3} & \bar{4} \\
\bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{0} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4}
\end{array}
$$

An effective means for proving that these two designs are not isomorphic is the computation of the statistics of 3-wise block intersection. It is enough to see that for $\mathcal{D}_1$ there is no triple of blocks that intersect in 0 points, while for $\mathcal{D}_2$ the computer found 119 such triples. This completes the proof of Theorem 1.1. □

We have used the computer to find some other properties of the two new symmetric designs constructed here. We got the following results:

THEOREM 3.1. *The 2-rank of both $\mathcal{D}_1$ and $\mathcal{D}_2$ equals* 114.

THEOREM 3.2. $Aut\mathcal{D}_1 \cong Aut\mathcal{D}_2 \cong (Z_{17}.Z_8) \times (Z_7.Z_3)$.

PROOF. It has shown out that an additional automorphism χ of order 3 acts on $\mathcal{D}_1$ and $\mathcal{D}_2$. χ commutes with ρ and σ but not with τ. □

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