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PARABOLIC INDUCTION AND JACQUET MODULES OF REPRESENTATIONS OF $O(2n, F)$

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ABSTRACT. For the sum of the Grothendieck groups of the categories of smooth finite length representations of $O(2n, F)$ (resp., $SO(2n, F)$), $n \geq 0$, (F a p-adic field), the structure of a module and a comodule over the sum of the Grothendieck groups of the categories of smooth finite length representations of $GL(n, F)$, $n \geq 0$, is achieved. The multiplication is defined in terms of parabolic induction, and the comultiplicitation in terms of Jacquet modules. Also, for even orthogonal groups, the combinatorial formula, which connects the module and the comodule structures, is obtained.

1. INTRODUCTION

In this paper, we deal with

$$R(O) = \bigoplus_{n \geq 0} R_n(O)$$

where $R_n(O)$ denotes the Grothendieck group of the category of smooth finite length representations of $O(2n, F)$, F a p-adic field. $R(O)$ is a module and a comodule over the Hopf algebra $R = \bigoplus_{n\geq 0} R_n$; here R_n denotes the Grothendieck group of the category of smooth finite length representations of $GL(n, F)$.

The structure of R was described by Zelevinsky in [Z1]. The definition of the multiplication $m : R \otimes R \to R$ and the comultiplication $m^* : R \to R \otimes R$ is based on the fact that for $0 \leq k \leq n$ there exists a standard parabolic subgroup of $GL(n, F)$ whose Levi factor is isomorphic to $GL(k, F) \times$

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$GL(n-k, F)$. The multiplication is defined using parabolic induction, and the comultiplication by Jacquet modules (see the third section of this paper). The structure of a Hopf algebra on R includes the Hopf axiom; it is the property that m^* is a ring homomorphism.

For $O(2n, F)$ and $0 \leq k \leq n$, there is a standard parabolic subgroup whose Levi factor is isomorphic to $GL(k, F) \times O(2(n-k), F)$. So, there is a natural way to define (using parabolic induction) the action $\rtimes$ of R on $R(O)$, and (using Jacquet modules) the mapping $\mu^* : R(O) \to R \times R(O)$. This is done in the sixth section.

There is also a connection between the module and the comodule structures on $R(O)$. Let

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*,$$

where $s : R \otimes R \to R \otimes R$ is the homomorphism determined by $s(r_1 \otimes r_2) = r_2 \otimes r_1$, $r_1, r_2 \in R$. Then we have

$$(*) \qquad \mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma),$$

so $R(O)$ is an M^*-Hopf module over R (see [T1] for the definition). The formula (*) can be used to find a composition series for Jacquet modules of parabolically induced representations.

The kind of work we have done for even orthogonal groups was first done by Tadić; in [T1] he introduced such a structure in the cases of symplectic and special odd-orthogonal groups, and he proved the combinatorial formula (*) for those groups. He also raised the question of the existence of such a structure for other series of classical p-adic groups.

We now give a short summary of the paper. In the second section, we give the definitions and some results of Bernstein and Zelevinsky, and Casselman, about parabolic induction and Jacquet modules. The third section describes the structure of R, as it is done in [Z1]. The fourth section is about standard parabolic subgroups of $SO(2n, F)$ and about $R(S)$ (the definition is analogous to $R(O)$). $R(S)$ is an R-module and R-comodule. The fifth section contains calculations in the root system for the case of D_n, i.e., for the group $SO(2n, F)$. This is used in the sixth section to find double cosets of $O(2n, F)$. In this section we also define the module and the comodule structures for even orthogonal groups. In the seventh section, we have applied the proof of the combinatorial formula from [T1] to our case.

I would like to close the introduction by thanking Marko Tadić, who suggested this project and helped its realisation. Also, I am very grateful to the referee for his valuable comments and English corrections.

2. Preliminaries

In this section, we shall introduce some basic notation and recall some results that will be needed in the rest of the paper. Our presentation follows the papers [BZ2] and [C].

A Hausdorff topological group G is called an *l-group* if any neighbourhood of the identity contains an open compact subgroup.

Let G be an l-group, M, U closed subgroups, such that M normalises U, $M \cap U = \{e\}$ and the subgroup $P = MU \subseteq G$ is closed; let θ be a character of U normalised by M. In such a situation, we define the functors

$$\begin{aligned} I_{U,\theta}, \qquad i_{U,\theta} &: AlgM \to AlgG, \\ r_{U,\theta} &: AlgG \to AlgM. \end{aligned}$$

(Here $AlgG$ denotes the category of algebraic (=smooth) representations of G.)

(a) Let $(\rho, L) \in AlgM$. Denote by $I(L)$ the space of functions $f : G \to L$ satisfying the following conditions:

1. $f(umg) = \theta(u)\ \Delta_U^{1/2}(m)\rho(m)(f(g)), \qquad u \in U,\ m \in M,\ g \in G.$
   (Here Δ_U denotes the modular character.)
2. There exists an open subgroup $K_f \subset G$ such that

$$f(gk) = f(g), \quad \text{for } g \in G,\ k \in K_f.$$

Define the representation $(\delta, I(L)) \in AlgG$ by $(\delta(g)f)(g') = f(g'g)$. We call δ an induced representation and denote it by $I_{U,\theta}(\rho)$.

Denote by $i(L)$ the subspace of $I(L)$ consisting of all functions compactly supported modulo the subgroup $P = MU$. The restriction of δ to the space $i(L)$ is called compactly induced and is denoted by $i_{U,\theta}(\rho)$.

(b) Let $(\pi, E) \in AlgG$. Denote by $E(U, \theta) \subseteq E$ the subspace spanned by the vectors of the form

$$\pi(u)\xi - \theta(u)\xi, \qquad u \in U,\ \xi \in E.$$

The quotient space $E/E(U, \theta)$ is called the θ-localisation of the space E and is denoted by $r_{U,\theta}(E)$. Define the representation $(\delta, r_{U,\theta}(E)) \in AlgM$ by

$$\delta(m)(\xi + E(U,\theta)) = \Delta_U^{-1/2}(m)(\pi(m)\xi + E(U,\theta)), \qquad m \in M,\ \xi \in E;$$

it is easily verified that δ is well-defined. Call the representation δ the θ-localisation of π and denote it by $r_{U,\theta}(\pi)$.

We shall now state a result of Bernstein and Zelevinsky (Theorem 5.2 of [BZ2]).

Let G be an l-group, P, M, U and Q, N, V be closed subgroups, θ be a character of U and ψ be a character of V. Suppose that

(1) $MU = P$, $NV = Q$, $M \cap U = N \cap V = \{e\}$, M normalises U and θ, N normalises V and ψ.

Then there are defined functors

$$i_{U,\theta} : AlgM \to AlgG \quad \text{and} \quad r_{V,\psi} : AlgG \to AlgN.$$

We want to compute the functor

$$F = r_{V,\psi} \circ i_{U,\theta} : AlgM \to AlgN.$$

It requires some complementary conditions. Suppose that

(2) The group G is countable in infinity, and U, V are limits of compact subgroups.

Consider the space $X = P \backslash G$ with its quotient-topology and the action δ of G on X defined by

$$\delta(g)(Ph) = Phg^{-1}, \quad g, h \in G, Ph \in X.$$

Suppose that

(3) The subgroup Q has a finite number of orbits on X. Acording to ([BZ1], 1.5), one can choose a numbering $Z_1, ..., Z_k$ of the Q-orbits on X such that all sets

$$Y_1 = Z_1, \quad Y_2 = Z_1 \cup Z_2, ..., \quad Y_k = Z_1 \cup ... \cup Z_k = X$$

are open in X. In particular, all Q-orbits on X are locally closed.

Fix a Q-orbit $Z \subseteq X$. Choose $\bar{w} \in G$ such that $P\bar{w}^{-1} \in Z$ and denote by w the corresponding inner automorphism of G: $w(g) = \bar{w}g\bar{w}^{-1}$. Call a subgroup $H \subseteq G$ decomposable with respect to the pair (M, U), if $H \cap (MU) = (H \cap M)(H \cap U)$. Suppose that

(4) The groups $w(P), w(M)$ and $w(U)$ are decomposable with respect to (N, V); the groups $w^{-1}(Q), w^{-1}(N)$ and $w^{-1}(V)$ are decomposable with respect to (M, U).

If the conditions (1)-(4) hold, we define the functor $\Phi_Z : AlgM \to AlgN$. Consider the condition

(*) The characters $w(\theta)$ and ψ coincide when restricted to the subgroup $w(U) \cap V$.

If (*) does not hold, set $\Phi_Z = 0$. If (*) holds then define the functor Φ_Z in the following way.

Set

$$\begin{aligned} M' &= M \cap w^{-1}(N), \quad N' = w(M') = w(M) \cap N, \\ V' &= M \cap w^{-1}(V), \quad \psi' = w^{-1}(\psi) \mid_{V'}, \\ U' &= N \cap w(U), \quad \theta' = w(\theta) \mid_{U'} . \end{aligned}$$

It is clear that the following functors are defined

$$\begin{aligned} r_{V',\psi'} &: AlgM \to AlgM', \\ w &: AlgM' \to AlgN', \\ i_{U',\theta'} &: AlgN' \to AlgN. \end{aligned}$$

Let $\varepsilon_1 = \Delta_U^{-1/2}\Delta_{U\cap w^{-1}(Q)}^{1/2}$ be a character of M', $\varepsilon_2 = \Delta_V^{-1/2}\Delta_{V\cap w(P)}^{1/2}$ be a character of N' and $\varepsilon = \varepsilon_1.w(\varepsilon_2)$ be a character of M'. We define Φ_Z by

$$\Phi_Z = i_{U,\theta'} \circ w \circ \varepsilon \circ r_{V',\psi'} : AlgM \to AlgN$$

(here ε is considered as a functor, *see* [BZ2] 1.5). In a more symmetric form,

$$\Phi_Z = i_{U',\theta'} \circ \varepsilon_2 \circ w \circ \varepsilon_1 \circ r_{V',\psi'}.$$

THEOREM 2.1. *Under the conditions (1)-(4) the functor $F = r_{V,\psi} \circ i_{U,\theta} : AlgM \to AlgN$ is glued from the functors Φ_Z where Z runs through all Q-orbits on X. More precisely, if orbits $Z_1, ..., Z_k$ are numerated so that all sets $Y_i = Z_1 \cup \ldots \cup Z_i$ $(i = 1, ..., k)$ are open in X, then there exists a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_k = F$ such that $F_i/F_{i+1} \simeq \Phi_{Z_i}$.*

(Let $\mathcal{A}$ be an abelian category and $C_1, C_2, \ldots, C_k \in \mathcal{A}$. We say that the object $D \in \mathcal{A}$ is glued from $C_1, C_2, \ldots, C_k$ if there is a filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_k = D$ in D, such that the set of quotients $\{D_i/D_{i-1}\}$ is isomorphic after a permutation to the set $\{C_i\}$.)

Let F be a locally compact nonarchimedean field. By an *algebraic F-group* we mean the group of F-points of some algebraic group, defined over F. In a natural locally compact topology such groups are l-groups.

Let G be a connected (in an algebraic sense) reductive F-group. Fix from now on a minimal parabolic subgroup $P_0 \subset G$ and a maximal split torus $A_0 \subset P_0$.

Let P be a parabolic subgroup containing P_0. We call such a group a standard parabolic subgroup. Let U be the unipotent radical of P. There exists a unique Levi subgroup in P containing A_0; denote it by M (it is a connected reductive F-group). It is known that P normalises U and has the Levi decomposition $P = MU$, $M \cap U = \{e\}$. We define the functors

$$i_{G,M} : AlgM \to AlgG \qquad \text{and} \qquad r_{M,G} : AlgG \to AlgM$$

by

$$i_{G,M} = i_{U,1}, \qquad r_{M,G} = r_{U,1}.$$

For $\sigma \in AlgM$ we call $i_{G,M}(\sigma)$ the parabolically induced representation of G by σ from P, and for $\pi \in AlgG$ we call $r_{M,G}(\pi)$ the Jacquet module of π with respect to P.

Denote by Σ the set of (reduced) roots of G relative to A_0. The choice of P_0 determines a basis Δ of Σ (which consists of simple roots). It also determines a set of positive roots Σ^+. Denote by W the Weyl group of G. For $\theta \subseteq \Delta$, we denote by W_θ the subgroup of W generated by all reflections $\{w_\alpha \mid \alpha \in \theta\}$. If $P = P_\theta = MU$ is the standard parabolic subgroup of G determined by θ, then W_θ is also denoted by W_M.

Let $\Omega,\ \theta \subset \Delta$. Now, we shall describe the set $[W_\theta \backslash W / W_\Omega]$, a set of representatives of $W_\theta \backslash W / W_\Omega$, defined in [C].

For $\alpha \in \Delta$, set

$$W^\alpha = \{w \in W \mid w\alpha > 0\}, \qquad {}^\alpha W = \{w \in W \mid w^{-1}\alpha > 0\}.$$

We have

$$\begin{aligned} [W/W_\Omega] &= \bigcap_{\alpha \in \Omega} W^\alpha, \qquad [W_\Omega \backslash W] = \bigcap_{\alpha \in \Omega} {}^\alpha W, \\ [W_\theta \backslash W / W_\Omega] &= [W_\theta \backslash W] \cap [W/W_\Omega]. \end{aligned}$$

If $P = P_\theta = MU$ and $Q = P_\Omega = NV$ are standard parabolic subgroups of G, then we have a bijection $W_M \backslash W / W_N \cong P \backslash G / Q$ (see [BT], 5.15,5.20). From this relation and Theorem 1.1 Bernstein and Zelevinsky obtained the geometric lemma ([BZ2]). The same result was obtained independently by Casselman in [C].

THEOREM 2.2 (Geometric lemma). *Let G be a connected reductive p-adic group, $P = P_\theta = MU$, $Q = P_\Omega = NV$ parabolic subgroups. Let σ be an admissible representation of M. Then $r_{N,G} \circ i_{G,M}(\sigma)$ has a composition series withfactors*

$$i_{N,N'} \circ w^{-1} \circ r_{M',M}(\sigma)$$

where $M' = M \cap w(N)$, $N' = w^{-1}(M) \cap N$ and $w \in [W_\theta \backslash W / W_\Omega]$.

Let π be a smooth finite length representation of G. We identify it canonically with an element of the Grothendieck group of the category of all smooth finite length representations of G. We denote this element by $s.s.(\pi)$ and call this map semi-simplification.

3. General linear group

In this section, we shall recall some results of the representation theory of p-adic general linear groups. The proofs can be found in [BZ2] and [Z1].

Fix the minimal parabolic subgroup of $GL(n,F)$ which consists of all upper triangular matrices in $GL(n,F)$. The standard parabolic subgroups of $GL(n,F)$ can be parametrized by ordered partitions of n: for $\alpha = (n_1, \dots, n_k)$ there exists a standard parabolic subgroup (denote it in this section by P_α) of $GL(n,F)$ whose Levi factor M_α is naturally isomorphic to $GL(n_1,F) \times \cdots \times GL(n_k,F)$.

Denote by R_n the Grothendieck group of the category of smooth representations of $GL(n,F)$. R_n is a free abelian group; it has a basis consisting of equivalence classes of irreducible smooth representations of $GL(n,F)$. Let

$$R = \bigoplus_{n\geq 0} R_n.$$

We shall define a multiplication and a comultiplication on R. Let π_1, π_2 be admissible representations of $GL(n_1,F), GL(n_2,F)$, resp., $n_1+n_2=n$. Define

$$\pi_1 \times \pi_2 = i_{GL(n,F),M(n_1,n_2)}(\pi_1 \otimes \pi_2).$$

Now, for irreducible smooth representations $\pi, \tau \in R$, we put $\pi \times \tau = s.s.(\pi \times \tau)$. We extend $\times$ $\mathbb{Z}$- bilinearly to $R \times R$. The induced mapping $R \otimes R \to R$, $\pi \otimes \tau \mapsto \pi \times \tau$ is denoted by m.

Let π be a smooth representation of $GL(n,F)$ of finite length. For $\alpha = (n_1, \dots, n_k)$ we define

$$r_{\alpha,(n)}(\pi) = r_{M_\alpha, GL(n,F)}(\pi).$$

This is a representation of $M_\alpha \cong GL(n_1,F) \times \cdots \times GL(n_k,F)$, so we may consider $s.s.(r_{\alpha,(n)}(\pi)) \in R_{n_1} \otimes \cdots \otimes R_{n_k}$. Now we define

$$m^*(\pi) = \sum_{k=0}^{n} s.s.(r_{(k,n-k),(n)}(\pi)) \in R \otimes R.$$

We extend m^* $\mathbb{Z}$-linearly to all R.

With the multiplication m and the comultiplication m^*, R is a graded Hopf algebra. This means that R is $\mathbb{Z}_+$-graded as an abelian group, m and m^* are $\mathbb{Z}_+$-graded, R has an algebra and coalgebra structure, and the comultiplication $m^* : R \to R \otimes R$ is a ring homomorphism.

Let $g \in GL(n,F)$. We denote by ${}^t g$ the transposed matrix of g , and by ${}^\tau g$ the matrix of g transposed with respect to the second diagonal.

4. Special orthogonal group $SO(2n,F)$

From now on, F will be a fixed local non-archimedean field of characteristic different from two.

The special orthogonal group $SO(2n,F)$, $n \geq 1$, is the group

$$SO(2n,F) = \{X \in SL(2n,F) \mid {}^\tau X X = I_{2n}\}.$$

For $n = 1$, we get

$$SO(2,F) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \middle| \lambda \in F^\times \right\} \cong F^\times.$$

$SO(0,F)$ is defined to be the trivial group.

Denote by A_0 the maximal split torus in $SO(2n,F)$ which consists of all diagonal matrices in $SO(2n,F)$. Hence,

$$A_0 = \left\{ diag(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) \middle| x_i \in F^\times \right\} \cong (F^\times)^n.$$

Denote by a the natural isomorphism of $(F^\times)^n$ to A_0 defined by $a(x_1, ..., x_n) = diag(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1})$.

We fix the minimal parabolic subgroup P_0 which consists of all upper triangular matrices in $SO(2n,F)$.

The root system is of type D_n :

the roots:	$\pm e_i \pm e_j,$	$1 \le i < j \le n,$	
the positive roots:	$e_i - e_j,$	$1 \le i < j \le n,$	
	$e_i + e_j,$	$1 \le i < j \le n,$	
the simple roots:	$\alpha_i = e_i - e_{i+1},$	$1 \le i \le n-1,$	$\alpha_n = e_{n-1} + e_n.$

The set of simple roots is denoted by Δ. The action of the simple roots on A_0 is given by

$$\begin{aligned} \alpha_i(a(x_1, ..., x_n)) &= x_i x_{i+1}^{-1}, \quad 1 \le i \le n-1, \\ \alpha_n(a(x_1, ..., x_n)) &= x_{n-1} x_n. \end{aligned}$$

Let us describe the standard parabolic subgroup $P_\theta = M_\theta U_\theta,\ \theta \subseteq \Delta$. For $i = 1, ..., n$ we define

$$\Omega_i = \begin{cases} \Delta \backslash \{\alpha_i\}, & i \ne n-1, \\ \Delta \backslash \{\alpha_n, \alpha_{n-1}\}, & i = n-1. \end{cases}$$

For $i = 0$, we put $\Omega_0 = \Delta$. If θ can be written in the form $\theta = \bigcap_{i \in I} \Omega_i,\ I = \{i_1, ..., i_k\},\ i_1 < i_2 < ... < i_k$, then

$$M_\theta = \left\{ diag(g_1, ..., g_k, h, {}^\tau g_k^{-1}, ..., {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), \right. \\ \left. h \in SO(2(n-m), F) \right\},$$

where $n_1 = i_1, n_1 + n_2 = i_2, ..., n_1 + \cdots + n_k = i_k = m$. Put $\alpha = (n_1, ..., n_k)$. M_θ is also denoted by M_α. In this case we have

$$M_\theta \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2(n-m), F).$$

If θ cannot be written in such a form (this happens when $\alpha_{n-1} \notin \theta,\ \alpha_n \in \theta$), then we have

$$\begin{aligned} M_\theta &= s\, M_\Omega\, s^{-1}, \quad \text{where } \Omega = (\theta \backslash \{\alpha_n\}) \cup \{\alpha_{n-1}\}, \\ s &= \begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{bmatrix}. \end{aligned}$$

Note that the presentation of θ in the form $\theta = \bigcap_{i\in I} \Omega_i$ is not always unique. Namely, when $\alpha_{n-1} \notin I$ and $\alpha_n \notin I$, we may take $n-1 \in I,\ n \in I$ or $n-1 \in I, \quad n \notin I$. In that case we have

$$M_\theta = \{diag(g_1, ..., g_k, h, \ {}^\tau g_k^{-1}, ..., \ {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), \\ h = diag(x, x^{-1}),\ x \in F^\times\},$$

so we may consider

$$M_\theta \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times GL(1, F),$$

or

$$M_\theta \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2, F).$$

For us, it will be important that for any ordered partition $\alpha = (n_1, ..., n_k)$ of a non-negative integer $m \leq n$ we have a standard parabolic subgroup of $SO(2n, F)$ whose Levi factor M_α is isomomorphic to $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2(n-m), F)$.

Now, take smooth finite length representations π of $GL(n, F)$ and σ of $SO(2m, F)$. Let $P_{(n)} = M_{(n)}U_{(n)}$ be a standard parabolic subgroup of $SO(2(m+n), F)$. Hence, $M_{(n)} \cong GL(n, F) \times SO(2m, F)$, so $\pi \otimes \sigma$ can be taken as a representation of $M_{(n)}$.Define

$$\pi \rtimes \sigma = i_{M_{(n)}, SO(2(m+n), F)}(\pi \otimes \sigma).$$

PROPOSITION 4.1. *Let π, π_1 and π_2 be finite length smooth representations of the groups $GL(n, F)$, $GL(n_1, F)$ and $GL(n_2, F)$ respectively, and let σ be a finite length smooth representation of $SO(2m, F)$. Then*

(i) $\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$,

(ii) $(\pi \rtimes \sigma)^\sim \cong \tilde{\pi} \rtimes \tilde{\sigma}$.

(Here $\tilde{\pi}$ denotes the contragredient representation of π.)

PROOF. The proof is straightforward and follows from [BZ2], Proposition 2.3. □

Denote by $R_n(S)$ the Grothendieck group of the category of all finite length smooth representations of $SO(2n, F)$. Define

$$R(S) = \bigoplus_{n\geq 0} R_n(S).$$

The multiplication of representations $\rtimes$ we introduced above gives rise to a multiplication $\rtimes : R \times R(S) \to R(S)$. For irreducible smooth representations $\pi \in R$ and $\sigma \in R(S)$, we put

$$\pi \rtimes \sigma = s.s.(\pi \rtimes \sigma),$$

and extend $\rtimes$ $\mathbb{Z}$-bilinearly to $R \times R(S)$. Now, we can get a $\mathbb{Z}$-linear mapping, denote it by $\mu : R \otimes R(S) \to R(S)$, which satisfies $\mu(\pi \otimes \sigma) = s.s.(\pi \rtimes \sigma)$ for $\pi \in R$ and $\sigma \in R(S)$.

PROPOSITION 4.2. *$(R(S), \mu)$ is a $\mathbb{Z}_+$-graded module over R.*

PROOF. See [Sw] for the definition of a module over a Hopf algebra. We are interested in the property of associativity, i.e., that the following diagram commutes:

$$\begin{array}{ccc} R \otimes R \otimes R(S) & \xrightarrow{id\otimes\mu} & R \otimes R(S) \\ {\scriptstyle m\otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ R \otimes R(S) & \xrightarrow{\mu} & R(S). \end{array}$$

The proof of this property relies on the previous proposition. □

Let σ be a finite length smooth representation of $SO(2n, F)$. Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of a non-negative integer $m \leq n$. Define

$$s_{\alpha,(0)}(\sigma) = r_{M_\alpha, SO(2n,F)}(\sigma).$$

This is a representation of $M_\alpha \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2(n-m), F)$, so we may consider $s.s.(s_{\alpha,(0)}(\sigma)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(S)$. Now we shall define a $\mathbb{Z}$-linear mapping $\mu^* : R(S) \to R \otimes R(S)$. For an irreducible smooth representation $\sigma \in R(S)$, we define

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(s_{(k),(0)}(\sigma)).$$

We extend μ^* $\mathbb{Z}$-linearly to $\mu^* : R(S) \to R \otimes R(S)$.

PROPOSITION 4.3. *$(R(S), \mu^*)$ is $\mathbb{Z}_+$-graded $R-$comodule.*

PROOF. The definition of a comodule over a Hopf algebra can be found in [Sw]. We are interested in coassociativity, i.e., that the following diagram commutes:

$$\begin{array}{ccc} R(S) & \xrightarrow{\mu^*} & R \otimes R(S) \\ {\scriptstyle \mu^*}\downarrow & & \downarrow{\scriptstyle id\otimes\mu^*} \\ R \otimes R(S) & \xrightarrow{m^*\otimes id} & R \otimes R \otimes R(S). \end{array}$$

The proof follows from [BZ2], Prop.2.3. □

The above construction is analogous those Tadić did in [T1] for $Sp(n, F)$ and $SO(2n+1, F)$.

5. CALCULATIONS IN THE ROOT SYSTEM, THE CASE OF D_n

In this section we shall make the calculations in the Weyl group we need for the geometric lemma. Precisely, for $i_1, i_2 \in \{1, 2, \ldots, n\}$ we shall find $[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ and for $w \in [W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$, determine $\Omega_{i_1} \cap w(\Omega_{i_2})$.

First, we shall describe the Weyl group:

$$W \cong \{\pm 1\}^{n-1} \rtimes Sym(n),$$

where

$$\{\pm 1\}^{n-1} = \left\{ (\epsilon_1, ..., \epsilon_n) \in \{\pm 1\}^n \mid \prod_i \epsilon_i = 1 \right\}.$$

$Sym(n)$ acts on the roots $\pm e_i \pm e_j$ by permutations of the set $\{e_1, ..., e_n\}$, and $(\epsilon_1, ..., \epsilon_n)$ acts as sign changes (-1 in the i-th place of $\epsilon = (\epsilon_1, ..., \epsilon_n)$ denotes the interchange of e_i and $-e_i$). For $p \in Sym(n)$ and $(\epsilon_1, ..., \epsilon_n) \in \{\pm 1\}^{n-1}$, we have

$$p(\epsilon_1, ..., \epsilon_n)p^{-1} = (\epsilon_{p^{-1}(1)}, ..., \epsilon_{p^{-1}(n)}).$$

It follows that

$$\begin{aligned} [p(\epsilon_1, ..., \epsilon_n)]^{-1} &= p^{-1}(\epsilon_{p^{-1}(1)}, ..., \epsilon_{p^{-1}(n)}), \\ [(\epsilon_1, ..., \epsilon_n)p]^{-1} &= (\epsilon_{p(1)}, ..., \epsilon_{p(n)})p^{-1}. \end{aligned}$$

Now we shall use the formulas from [C] for $[W_\Theta \backslash W / W_\Omega]$ we listed before. The beginning of our calculation is almost the same as in [T1], and the first four lemmas are very similar.

By the definition of the action of W on roots, for $p \in Sym(n)$ and $(\epsilon_1, ..., \epsilon_n) \in \{\pm 1\}^n$ we have

$$\begin{aligned} p\epsilon(\alpha_i) &= p\epsilon(e_i - e_{i+1}) = p(\epsilon_i e_i - \epsilon_{i+1} e_{i+1}) \\ &= \epsilon_i e_{p(i)} - \epsilon_{i+1} e_{p(i+1)}, \quad 1 \le i \le n-1, \\ p\epsilon(\alpha_n) &= p\epsilon(e_{n-1} + e_n) = p(\epsilon_{n-1} e_{n-1} + \epsilon_n e_n) = \epsilon_{n-1} e_{p(n-1)} + \epsilon_n e_{p(n)}. \end{aligned}$$

As we said, $W^{\alpha_i} = \{w \in W \mid w\alpha_i > 0\}$. If we check when $p\epsilon(\alpha_i) > 0$, $1 \le i \le n$, then we easily get the following lemma.

LEMMA 5.1. *a) For $1 \le i \le n-1$, W^{α_i} is the disjoint union of the following three sets:*

(i) $\{p\epsilon \in W \mid \epsilon_i = \epsilon_{i+1} = 1,\ p(i) < p(i+1)\}$;
(ii) $\{p\epsilon \in W \mid \epsilon_i = 1,\ \epsilon_{i+1} = -1\}$;
(iii) $\{p\epsilon \in W \mid \epsilon_i = \epsilon_{i+1} = -1,\ p(i) > p(i+1)\}$

b) W^{α_n} is the disjoint union of the following three sets:

(i) $\{p\epsilon \in W \mid \epsilon_{n-1} = \epsilon_n = 1\}$;
(ii) $\{p\epsilon \in W \mid \epsilon_{n-1} = 1,\ \epsilon_n = -1,\ p(n-1) < p(n)\}$;
(iii) $\{p\epsilon \in W \mid \epsilon_{n-1} = -1,\ \epsilon_n = 1,\ p(n-1) > p(n)\}$.

In the same way, we can compute ${}^{\alpha_i}W = \{w \in W \mid w^{-1}\alpha_i > 0\}$.

LEMMA 5.2. *a) For $1 \le i \le n-1$, ${}^{\alpha_i}W$ is the disjoint union of the following three sets:*

(i) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(i)} = \epsilon_{p^{-1}(i+1)} = 1,\ p^{-1}(i) < p^{-1}(i+1)\}$;
(ii) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(i)} = 1,\ \epsilon_{p^{-1}(i+1)} = -1\}$;
(iii) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(i)} = \epsilon_{p^{-1}(i+1)} = -1,\ p^{-1}(i) > p^{-1}(i+1)\}$.

b) ${}^{\alpha_n}W$ *is the disjoint union of the following three sets:*

(i) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(n-1)} = \epsilon_{p^{-1}(n)} = 1\}$;

(ii) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(n-1)} = 1,\ \epsilon_{p^{-1}(n)} = -1,\ p^{-1}(n-1) < p^{-1}(n)\}$;

(iii) $\{p\epsilon \in W \mid \epsilon_{p^{-1}(n-1)} = -1,\ \epsilon_{p^{-1}(n)} = 1,\ p^{-1}(n-1) > p^{-1}(n)\}$.

In the next lemma, we shall use the formula $[W/W_\Omega] = \bigcap_{\alpha\in\Omega} W^\alpha$, for $\Omega \subseteq \Delta$.

LEMMA 5.3. *Let $1 \le i \le n$ and let $0 \le j \le i$. Denote by Y_j^i the set of all $p\epsilon \in W$ such that the following six conditions are satisfied:*

(i) $\epsilon_k = 1,$ *for $1 \le k \le j$;*

(ii) $p(k_1) < p(k_2),$ *for $1 \le k_1 < k_2 \le j$;*

(iii) $\epsilon_k = -1,$ *for $j+1 \le k \le i$;*

(iv) $p(k_1) > p(k_2),$ *for $j+1 \le k_1 < k_2 \le i$;*

(v) $\epsilon_k = 1,$ *for $i+1 \le k \le n-1$;*

(vi) $p(k_1) < p(k_2),$ *for $i+1 \le k_1 < k_2 \le n$.*

Denote by $\bar{Y}_j^n$ the set of all $p\epsilon \in W$ which satisfy the same conditions (for $i = n$), but instead of (iii), the condition

(iii') $\epsilon_k = -1,$ *for $j+1 \le k \le n-1,$* $\epsilon_n = 1.$

Then

$$[W/W_{\Omega_i}] = \bigcup_{0\le j\le i} Y_j^i,$$

$$[W/W_{\bar{\Omega}_n}] = \bigcup_{0\le j\le n-1} \bar{Y}_j^n.$$

Here $\bar{\Omega}_n = \Delta\backslash\{\alpha_{n-1}\}$.

PROOF. Take $p\epsilon \in [W/W_{\Omega_i}] = \bigcap_{\alpha\in\Omega_i} W^\alpha$. If $i < n-1$, then $p\epsilon \in W^{\alpha_n} \cap W^{\alpha_{n-1}}$. FromLemma 5.1 a) for $n-1$ and Lemma 5.1 b) for n we get

$$\epsilon_{n-1} = 1,\ \epsilon_n = -1, \quad p(n-1) < p(n),$$

or

$$\epsilon_{n-1} = 1,\ \epsilon_n = 1, \quad p(n-1) < p(n).$$

Anyway, $\epsilon_{n-1} = 1,\ p(n-1) < p(n)$. Further, Lemma 5.1.a) implies

$$\epsilon_{i+1} = \epsilon_{i+2} = \cdots = \epsilon_{n-1} = 1, \quad p(i+1) < p(i+2) < \cdots < p(n).$$

Now, we have

$$\begin{aligned} &\epsilon_k = 1, \quad \text{for } i+1 \le k \le n-1, \text{ and} \\ &p(k_1) < p(k_2), \quad \text{for } i+1 \le k_1 < k_2 \le n. \end{aligned}$$

This condition is also satisfied for $i = n-1$ or $i = n$, because in those cases it is empty.

Since $p\epsilon \in W^{\alpha_k}, \forall k \in \{1,\dots,i-1\}$, Lemma 5.1 implies that for any $k \in \{1,\dots,i-1\}$, we have $\epsilon_k = \epsilon_{k+1} = 1$ or $\epsilon_k = 1$, $\epsilon_{k+1} = -1$ or $\epsilon_k = \epsilon_{k+1} = -1$. We cannot have $\epsilon_k = -1$, $\epsilon_{k+1} = 1$. So we conclude that there exists $j \in \{0,1,...,i\}$ such that $\epsilon_j = 1$ for $1 \le k \le j-1$ and $\epsilon_k = -1$ for $j+1 \le k \le i-1$. Lemma 5.1 also implies $p(k) < p(k+1)$ for $1 \le k \le j-1$ and $p(k) > p(k+1)$ for $j+1 \le k \le i-1$. Hence, $p\epsilon \in Y_j^i$ where $0 \le j \le i$.

If $p\epsilon \in \bigcup_{0\le j\le i} Y_j^i$,then we see from Lemma 5.1 that $p\epsilon \in W^{\alpha_l}$ for $l \ne i$, in the case $i \ne n-1$, and $p\epsilon \in W^{\alpha_l}$ for $l \ne n-1$, $l \ne n$ in the case $i = n-1$. This proves the other inclusion.

Let $p\epsilon \in [W/W_{\bar{\Omega}_n}] = \bigcap_{l\ne n-1} W^{\alpha_l}$.

Suppose that $\epsilon_{n-1} = 1$. Then by Lemma 5.1 a), we get $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{n-1} = 1$, $p(k_1) < p(k_2)$, for $1 \le k_1 < k_2 \le n-1$. The condition $\prod_i \epsilon_i = 1$ gives us $\epsilon_n = 1$. Put $j = n-1$. Then, the conditions $(i), (ii)$ and (iii') are satisfied, and the others are empty.

Let $\epsilon_{n-1} = -1$. Then by Lemma 5.1 b) we get $\epsilon_n = 1$, $p(n-1) > p(n)$.It follows from Lemma 5.1.a) that there exists $j \in \{0,1,...,n-2\}$ such that

$$\epsilon_k = 1, \quad \text{for } 1 \le k \le j, \quad p(k_1) < p(k_2), \quad \text{for } 1 \le k_1 < k_2 \le j,$$

and

$$\begin{aligned}&\epsilon_k = -1, \quad \text{for } j+1 \le k \le n-1, \quad p(k_1) > p(k_2),\\ &\text{for } j+1 \le k_1 < k_2 \le n-1.\end{aligned}$$

Together with the first condition, we get

$$\epsilon_k = -1, \quad \text{for } j+1 \le k \le n-1, \quad p(k_1) > p(k_2), \quad \text{for } j+1 \le k_1 < k_2 \le n.$$

Therefore, the conditions $(i), (ii), (iii')$ and (iv) are satisfied, and the others are empty.

The other inclusion can be proved as before. □

REMARK 5.1. *If $p\epsilon \in W$, $\epsilon = (\epsilon_{1,\dots,}\epsilon_n)$, then $\prod_{i=1}^{n} \epsilon_i = 1$. Thus we have*

$$\begin{aligned}&\textit{for } i < n: && p\epsilon \in Y_j^i \textit{ implies } \epsilon_n = (-1)^{i-j},\\ &\textit{for } i = n: && \textit{if } n-j \textit{ odd, then } Y_j^n = \emptyset,\\ & && \textit{if } n-j > 0 \textit{ even, then } \bar{Y}_j^n = \emptyset.\end{aligned}$$

If $j = n$, then $\bar{Y}_n^n = \{id\} \subseteq \bar{Y}_{n-1}^n$, so we can write

$$[W/W_{\bar{\Omega}_n}] = \bigcup_{0\le j\le n} \bar{Y}_j^n.$$

For the set $[W_\Omega \backslash W]$, we can simply use the relation $[W/W_\Omega]^{-1} = [W_\Omega \backslash W]$ and the previous lemma to obtain the following:

LEMMA 5.4. *Let $1 \leq i \leq n$ and let $0 \leq j \leq i$. Denote by X_j^i the set of all $p\epsilon \in W$ such that the following six conditions are satisfied:*

(i) $\epsilon_{p^{-1}(k)} = 1$, *for* $1 \leq k \leq j$;
(ii) $p^{-1}(k_1) < p^{-1}(k_2)$, *for* $1 \leq k_1 < k_2 \leq j$;
(iii) $\epsilon_{p^{-1}(k)} = -1$, *for* $j+1 \leq k \leq i$;
(iv) $p^{-1}(k_1) > p^{-1}(k_2)$, *for* $j+1 \leq k_1 < k_2 \leq i$;
(v) $\epsilon_{p^{-1}(k)} = 1$, *for* $i+1 \leq k \leq n-1$;
(vi) $p^{-1}(k_1) < p^{-1}(k_2)$, *for* $i+1 \leq k_1 < k_2 \leq n$.

Denote by $\tilde{X}_j^n$ the set of all $p\epsilon \in W$ which satisfy the same conditions (for $i = n$), but instead of (iii), the condition

(iii') $\epsilon_{p^{-1}(k)} = -1$, *for* $j+1 \leq k \leq n-1$, $\epsilon_{p^{-1}(n)} = 1$.

Then,

$$[W_{\Omega_i} \backslash W] = \bigcup_{0 \leq j \leq i} X_j^i,$$

$$[W_{\tilde{\Omega}_n} \backslash W] = \bigcup_{0 \leq j \leq n-1} \tilde{X}_j^n.$$

□

Let $i_1, i_2 \in \{1, ..., n\}$. For integers d, k such that

$$0 \leq d \leq min\{i_1, i_2\},$$

$$max\{0, (i_1 + i_2 - n) - d\} \leq k \leq min\{i_1, i_2\} - d,$$

we define a permutation $p_n(d,k)_{i_1,i_2}$ in the same way as in [T1]:

$$p_n(d,k)_{i_1,i_2}(j) = \begin{cases} j, & \text{for } 1 \leq j \leq k; \\ j + i_1 - k, & \text{for } k+1 \leq j \leq i_2 - d; \\ (i_1 + i_2 - d + 1) - j, & \text{for } i_2 - d + 1 \leq j \leq i_2; \\ j - i_2 + k, & \text{for } i_2 + 1 \leq j \\ & \leq i_1 + i_2 - d - k; \\ j, & \text{for } i_1 + i_2 - d - k + 1 \\ & \leq j \leq n. \end{cases}$$

The conditions on d and k imply that $p = p_n(d,k)_{i_1,i_2}$ is well-defined. For $k \geq 0$, we set

$$\mathbf{1}_k = \underbrace{1, 1, ..., 1}_{k \text{ times}} \quad \text{and} \quad -\mathbf{1}_k = \underbrace{-1, -1, ..., -1}_{k \text{ times}}.$$

a) If $i_1, i_2 \leq n$, $0 \leq d \leq min\{i_1, i_2\}$, d even, $max\{0, (i_1 + i_2 - n) - d\} \leq k \leq min\{i_1, i_2\} - d$, then we define

$$q_n(d,k)_{i_1,i_2}^{(0,0)} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2}).$$

If $i_1, i_2 < n$, $0 \le d \le min\{i_1, i_2\}$, d odd, $max\{0, (i_1+i_2-n)-d+1\} \le k \le min\{i_1, i_2\} - d$, then we define

$$q_n(d,k)^{(0,0)}_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2-1}, -1).$$

b) If $i_1, i_2 < n$, $0 \le d \le min\{i_1, i_2\}$, d even, $k = i_1 + i_2 - n - d \ge 0$, then we define

$$q_n(d,k)^{(1,1)}_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d-1}, -\mathbf{1}_{d+1}, \mathbf{1}_{n-i_2-1}, -1).$$

c) If $i_1 \le n$, $i_2 < n$, $0 \le d \le min\{i_1, i_2\}$, d odd, $k = i_1 + i_2 - n - d \ge 0$, then we define

$$q_n(d,k)^{(1,0)}_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2-1}, -1).$$

d) If $i_1 < n$, $i_2 \le n$, $0 \le d \le min\{i_1, i_2\}$, d odd, $k = i_1 + i_2 - n - d \ge 0$, then we define

$$q_n(d,k)^{(0,1)}_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d-1}, -\mathbf{1}_{d+1}, \mathbf{1}_{n-i_2}).$$

$q_n(d,k)^{(0,0)}_{i_1,i_2}, q_n(d,k)^{(1,1)}_{i_1,i_2}, q_n(d,k)^{(1,0)}_{i_1,i_2}$ and $q_n(d,k)^{(0,1)}_{i_1,i_2}$ are elements of W.

LEMMA 5.5. *Let $i_1, i_2 \in \{1, ..., n\}$. Suppose that integers j_1 and j_2 satisfy $0 \le j_1 \le i_1$ and $0 \le j_2 \le i_2$. If $X^{i_1}_{j_1} \cap Y^{i_2}_{j_2} \neq \emptyset$, then one of the following three conditions is satisfied:*

(i) $i_1 - j_1 = i_2 - j_2$;

(ii) $i_1 - j_1 = i_2 - j_2 + 1$ *even;*

(iii) $i_2 - j_2 = i_1 - j_1 + 1$ *even.*

In that case, we have:

(a) *If $i_1 - j_1 = i_2 - j_2$ is even, then* $X^{i_1}_{j_1} \cap Y^{i_2}_{j_2} =$

$$\left\{q_n(d,k)^{(0,0)}_{i_1,i_2} \mid d = i_1 - j_1, max\{0, (i_1+i_2-n)-d\} \le k \le min\{i_1, i_2\} - d\right\}$$

(b) *If $i_1 - j_1 = i_2 - j_2$ is odd, then* $X^{i_1}_{j_1} \cap Y^{i_2}_{j_2} =$

$$\left\{q_n(d,k)^{(0,0)}_{i_1,i_2} \mid d = i_1 - j_1, \right.$$
$$\left. max\{0, (i_1+i_2-n)-d+1\} \le k \le min\{i_1, i_2\} - d\right\}$$
$$\cup \left\{q_n(d,k)^{(1,1)}_{i_1,i_2} \mid d = i_1 - j_1 - 1, k = i_1 + i_2 - n - d \ge 0\right\}.$$

(c) *If $i_1 - j_1 = i_2 - j_2 + 1$ is even, then*

$$X^{i_1}_{j_1} \cap Y^{i_2}_{j_2} = \left\{q_n(d,k)^{(1,0)}_{i_1,i_2} \mid d = i_2 - j_2, k = i_1 + i_2 - n - d \ge 0\right\}.$$

(d) *If $i_2 - j_2 = i_1 - j_1 + 1$ is even, then*

$$X^{i_1}_{j_1} \cap Y^{i_2}_{j_2} = \left\{q_n(d,k)^{(0,1)}_{i_1,i_2} \mid d = i_1 - j_1, k = i_1 + i_2 - n - d \ge 0\right\}.$$

PROOF. Let $p\epsilon \in W$. Then $p\epsilon \in X_{j_1}^{i_1} \cap Y_{j_2}^{i_2}$ if and only if the following twelve conditions are satisfied:

(1) $\epsilon_{p^{-1}(l)} = 1$, for $1 \leq l \leq j_1$;
(2) $p^{-1}(l_1) < p^{-1}(l_2)$, for $1 \leq l_1 < l_2 \leq j_1$;
(3) $\epsilon_{p^{-1}(l)} = -1$, for $j_1 + 1 \leq l \leq i_1$;
(4) $p^{-1}(l_1) > p^{-1}(l_2)$, for $j_1 + 1 \leq l_1 < l_2 \leq i_1$;
(5) $\epsilon_{p^{-1}(l)} = 1$, for $i_1 + 1 \leq l \leq n - 1$;
(6) $p^{-1}(l_1) < p^{-1}(l_2)$, for $i_1 + 1 \leq l_1 < l_2 \leq n$;
(7) $\epsilon_l = 1$, for $1 \leq l \leq j_2$;
(8) $p(l_1) < p(l_2)$, for $1 \leq l_1 < l_2 \leq j_2$;
(9) $\epsilon_l = -1$, for $j_2 + 1 \leq l \leq i_2$;
(10) $p(l_1) > p(l_2)$, for $j_2 + 1 \leq l_1 < l_2 \leq i_2$;
(11) $\epsilon_l = 1$, for $i_2 + 1 \leq l \leq n - 1$;
(12) $p(l_1) < p(l_2)$, for $i_2 + 1 \leq l_1 < l_2 \leq n$.

Suppose that there exists $p\epsilon \in X_{j_1}^{i_1} \cap Y_{j_2}^{i_2}$. Then conditions (1),(3) and (5) give that the number of -1's which appear in ϵ must be $i_1 - j_1$ if $i_1 - j_1$ is even, or $i_1 - j_1 + 1$ if $i_1 - j_1$ is odd. Conditions (7),(9) and (11) give that the number of -1's which apear in ϵ must be $i_2 - j_2$ if $i_2 - j_2$ is even, or $i_2 - j_2 + 1$ if $i_2 - j_2$ is odd. We conclude that the difference between $i_1 - j_1$ and $i_2 - j_2$ is at most 1, and, if they are not equal, the bigger one is even. Thus, we get conditions $(i), (ii)$ and (iii) from the lemma.

a) If $i_1 - j_1 = i_2 - j_2$ even, then $\epsilon_n = 1, \quad \epsilon_{p^{-1}(n)} = 1$, so $p\epsilon$ satisfies conditions (1)-(12) from Lemma 4.5 [T1], which gives the statement.

b) Let $i_1 - j_1 = i_2 - j_2$ odd. If $i_1 = n$ or $i_2 = n$, then there is no $p\epsilon \in W$ which satisfies conditions (1)-(12), so $X_{j_1}^{i_1} \cap Y_{j_2}^{i_2} = \emptyset$. Suppose $i_1, i_2 < n$. From (7),(9) and (11), we conclude that

$$\epsilon = (\mathbf{1}_{j_2}, -\mathbf{1}_{i_2-j_2}, \mathbf{1}_{n-i_2-1}, -1).$$

Conditions (3),(7),(9) and (11) imply

$$p([j_2 + 1, i_2]_{\mathbb{N}} \cup \{n\}) = [j_1 + 1, i_1]_{\mathbb{N}} \cup \{n\}.$$

If $p(n) = n$, then conditions (1)-(12)restricted to the set $\{1, \dots, n-1\}$ are the same as in Lemma 4.5[T1]. It follows that $p = p_n(d,k)_{i_1,i_2}$ and $i_1 + i_2 - d - k + 1 \leq n$, i.e., $k \geq i_1 + i_2 - n - d + 1$.

If $p(n) \neq n$, then from (4) and (10) we see that

$$\begin{aligned} p(j_2 + 1) &= n, \\ p^{-1}(j_1 + 1) &= n, \\ p([j_2 + 2, i_2]_{\mathbb{N}}) &= [j_1 + 2, i_1]_{\mathbb{N}}. \end{aligned}$$

Set $d = i_2 - j_2 - 1$. By (10), p is order-reversing as a mapping $p : [j_2 + 2, i_2]_{\mathbb{N}} \to [j_1 + 2, i_1]_{\mathbb{N}}$. It follows that

$$p(j) = i_1 - (j - j_2 - 2) = (i_1 + i_2 - d + 1) - j, \quad \text{for} \quad i_2 - d + 1 \leq j \leq i_2.$$

From $p(j_2+1)=n$ we have $p^{-1}(n)=j_2+1$. Together with (6), this implies

$$p^{-1}([i_1+1,n-1]_{\mathbb{N}}) \subseteq [1,j_2]_{\mathbb{N}}.$$

In the same, way we get

$$p([i_2+1,n-1]_{\mathbb{N}}) \subseteq [1,j_1]_{\mathbb{N}}.$$

Let $K=p^{-1}([i_1+1,n-1]_{\mathbb{N}})$, $\quad L=[1,j_2]_{\mathbb{N}} \setminus K$. Suppose that $L \neq \emptyset$. Since $p(K \cup \{j_2+1\})=[i_1+1,n]_{\mathbb{N}}$, we have $p(K)>p(L)$, and from (8) we see that $K>L$, (i.e., $p>q$, $\quad \forall p \in K$, $\quad \forall q \in L$). Thus, there exists $k \in \{1,...,j_2\}$ such that

$$p^{-1}([i_1+1,n-1]_{\mathbb{N}})=[k+1,j_2]_{\mathbb{N}}.$$

If $L=\emptyset$, we put $k=0$, so the above condition is satisfied. Now, we have

$$(*) \qquad p^{-1}([i_1+1,n]_{\mathbb{N}})=[k+1,j_2+1]_{\mathbb{N}}=[k+1,i_2-d]_{\mathbb{N}},$$

$$\begin{aligned} n-i_1-1 &= i_2-d-k-1, \\ k &= i_1+i_2-n-d \geq 0. \end{aligned}$$

In the same way, we get

$$p([i_2+1,n]_{\mathbb{N}})=[k+1,i_1-d]_{\mathbb{N}}.$$

From (12), we obtain

$$p(j)=k+1+j-i_2-1=j-i_2+k, \qquad i_2+1 \leq j \leq n$$

By $(*)$, we have

$$p([k+1,i_2-d]_{\mathbb{N}})=[i_1+1,n]_{\mathbb{N}},$$

and from (6) we see that

$$p(j)=i_1+1+j-k-1=j+i_1-k, \qquad k+1 \leq j \leq i_2-d.$$

It remains to determine p on $[1,k]_{\mathbb{N}}$. From the above observations, we get

$$p([1,k]_{\mathbb{N}})=[1,k]_{\mathbb{N}},$$

so by (8), we have

$$p(j)=j, \qquad \text{for} \quad 1 \leq j \leq k.$$

We conclude that $p=p_n(d,k)_{i_1,i_2}$.

It remains to prove that $q=q_n(d,k)^{(0,0)}_{i_1,i_2} \in X^{i_1}_{j_1} \cap Y^{i_2}_{j_2}$ when $d=i_1-j_1=i_2-j_2$ and

$$\max\{0,(i_1+i_2-n)-d\} < k \leq min\{i_1,i_2\}-d,$$

and $q'=q_n(d,k)^{(1,1)}_{i_1,i_2} \in X^{i_1}_{j_1} \cap Y^{i_2}_{j_2}$ when $d=i_1-j_1-1=i_2-j_2-1$ and $k=i_1+i_2-n-d \geq 0$.

One sees directly from the definition of q and q' that conditions (7)-(12) are satisfied. In the same way, one sees that conditions (1)-(6) are satisfied.

c) Let $i_1 - j_1 = i_2 - j_2 + 1$ even. If $i_2 = n$, then there is no $p\epsilon \in W$ which satisfies conditions (1)-(12), so $X_{j_1}^{i_1} \cap Y_{j_2}^{i_2} = \emptyset$.

Suppose that $i_2 < n$. Set $d = i_2 - j_2$. From (1),(3),(5),(7) and (9), we see that

$$\begin{aligned} \epsilon_n &= -1, \quad \epsilon_{p^{-1}(n)} = 1, \\ \epsilon &= (\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2-1}, -1), \\ p([j_2+1, i_2]_{\mathbf{N}} \cup \{n\}) &= [j_1+1, i_1]_{\mathbf{N}}. \end{aligned}$$

From (4), we get

$$\begin{aligned} p^{-1}(j_1+1) &= n, \\ p([j_2+1, i_2]_{\mathbf{N}}) &= [j_1+2, i_1]_{\mathbf{N}}, \end{aligned}$$

$$p(j) = i_1 - (j - j_2 - 1) = (i_1 + i_2 - d + 1) - j, \qquad i_2 - d + 1 \leq j \leq i_2.$$

In the same way as in (b), it follows from $p(n) = j_1 + 1$ that

$$p([i_2+1, n-1]_{\mathbf{N}}) = [k+1, j_1]_{\mathbf{N}}, \qquad \text{where } k = i_1 + i_2 - n - d \geq 0,$$

and

$$p(j) = j - i_2 + k, \qquad i_2 + 1 \leq j \leq n.$$

We conclude that

$$p([1, i_2 - d]_{\mathbf{N}}) = [1, k]_{\mathbf{N}} \cup [i_1 + 1, n]_{\mathbf{N}}.$$

From (8), we have

$$\begin{aligned} p([1, k]_{\mathbf{N}}) &= [1, k]_{\mathbf{N}}, \\ p([k+1, i_2 - d]_{\mathbf{N}}) &= [i_1 + 1, n]_{\mathbf{N}}, \end{aligned}$$

and

$$\begin{aligned} p(j) &= j, \qquad 1 \leq j \leq k, \\ p(j) &= i_1 + 1 + j - k - 1 = j + i_1 - k, \qquad k + 1 \leq j \leq i_2 - d. \end{aligned}$$

Therefore, $p = p_n(d, k)_{i_1, i_2}$.

The rest of proof is same as in (b).

(d) Analogous to (c). □

For $i_1 = n, \quad i_2 \leq n, \quad 0 \leq d \leq i_2, \quad d$ odd, $k = i_2 - d$, we define

$$q_n(d, k)_{n, i_2}^{(-1,-1)} = p_n(d, k)_{i_1, i_2}(\mathbf{1}_{i_2-d+1}, -\mathbf{1}_{d-1}, \mathbf{1}_{n-i_2}).$$

For $i_1 = n, \quad i_2 < n, \quad 0 < d \leq i_2, \quad d$ even, $k = i_2 - d$, we define

$$q_n(d, k)_{n, i_2}^{(\pm 1,-1)} = p_n(d, k)_{i_1, i_2}(\mathbf{1}_{i_2-d+1}, -\mathbf{1}_{d-1}, \mathbf{1}_{n-i_2-1}, -1).$$

For $i_1 = n, \quad i_2 < n, \quad d = 0$, $k = i_2$, we define

$$q_n(d, k)_{n, i_2}^{(\pm 1,-1)} = p_n(d, k)_{i_1, i_2}.$$

For $i_1 = i_2 = n,\quad 0 < d \le n,\quad d$ even, $k = n - d$, we define

$$q_n(d,k)^{(-2,-2)}_{n,n} = p_n(d,k)_{n,n}(\mathbf{1}_{n-d+1}, -\mathbf{1}_{d-2}, 1).$$

For $i_1 = i_2 = n,\quad d = 0$, $k = n$, we define

$$q_n(d,k)^{(-2,-2)}_{n,n} = p_n(d,k)_{n,n}.$$

An argument analogous to that for Lemma 5.5 gives

LEMMA 5.6. *Let $i_2 \in \{1, ..., n\}$. Suppose that integers j_1 and j_2 satisfy $0 \le j_1 \le n-1$ and $1 \le j_2 \le i_2$. If $\bar{X}^n_{j_1} \cap Y^{i_2}_{j_2} \ne \emptyset$, then one of the following two conditions is satisfied:*

(i) *$(n-1) - j_1 = i_2 - j_2$ even,*
(ii) *$(n-1) - j_1 = i_2 - j_2 + 1$ even.*

In that case, we have:

(a) *If $(n-1) - j_1 = i_2 - j_2 > 0$ is even, then*

$$\bar{X}^n_{j_1} \cap Y^{i_2}_{j_2} = \left\{ q_n(d,k)^{(-1,-1)}_{n,i_2} \mid d = n - j_1, k = i_2 - d \right\},$$

and for $(n-1) - j_1 = i_2 - j_2 = 0$, we have $\bar{X}^n_{j_1} \cap Y^{i_2}_{j_2} =$

$$\left\{ q_n(d,k)^{(-1,-1)}_{n,i_2} \mid d = 1, k = i_2 - d \right\} \cup \left\{ q_n(d,k)^{(\pm 1,-1)}_{n,i_2} \mid d = 0, k = i_2 - d \right\}.$$

(b) *If $(n-1) - j_1 = i_2 - j_2 + 1$ is even, then $n - 1 - j_1 > 0$ and*

$$\bar{X}^n_{j_1} \cap Y^{i_2}_{j_2} = \left\{ q_n(d,k)^{(\pm 1,-1)}_{n,i_2} \mid d = n - j_1 - 1, k = i_2 - d \right\}.$$

PROPOSITION 5.7. *Let $i_1, i_2 \in \{1, ..., n\}$. Then, $\left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right] =$*

$$= \left[\bigcup_{\substack{0 \le d \le \\ \min\{i_1,i_2\} \\ d-even}} \left(\begin{array}{r} \left\{ q_n(d,k)^{(0,0)}_{i_1,i_2} \mid \max\{0, (i_1 + i_2 - n - d)\} \le k \right. \\ \left. \le \min\{i_1,i_2\} - d \right\} \\ \cup \left\{ q_n(d,k)^{(1,1)}_{i_1,i_2} \mid k = i_1 + i_2 - n - d \ge 0 \right\} \end{array} \right) \right]$$

$$\cup \left[\bigcup_{\substack{0 \le d \le \\ \min\{i_1,i_2\} \\ d-odd}} \left(\begin{array}{r} \left\{ q_n(d,k)^{(0,0)}_{i_1,i_2} \mid \max\{0, (i_1 + i_2 - n - d) + 1\} \right. \\ \left. \le k \le \min\{i_1,i_2\} - d \right\} \\ \cup \left\{ q_n(d,k)^{(1,0)}_{i_1,i_2} \mid k = i_1 + i_2 - n - d \ge 0 \right\} \\ \cup \left\{ q_n(d,k)^{(0,1)}_{i_1,i_2} \mid k = i_1 + i_2 - n - d \ge 0 \right\} \end{array} \right) \right]$$

Particularly:

(a) If $i_1 = n, \quad i_2 < n$, then

$$
\begin{aligned}
[W_{\Omega_n}\backslash W/W_{\Omega_{i_2}}] &= \left[\bigcup\nolimits_{\substack{0\le d\le i_2\\ d-even}}\left\{q_n(d,k)^{(0,0)}_{n,i_2} \mid k = i_2 - d\right\}\right]\\
&\cup \left[\bigcup\nolimits_{\substack{0\le d\le i_2\\ d-odd}}\left\{q_n(d,k)^{(1,0)}_{n,i_2} \mid k = i_2 - d\right\}\right].
\end{aligned}
$$

(b) If $i_1 < n, \quad i_2 = n$, then

$$
\begin{aligned}
[W_{\Omega_{i_1}}\backslash W/W_{\Omega_n}] &= \left[\bigcup\nolimits_{\substack{0\le d\le i_1\\ d-even}}\left\{q_n(d,k)^{(0,0)}_{i_1,n} \mid k = i_1 - d\right\}\right]\\
&\cup \left[\bigcup\nolimits_{\substack{0\le d\le i_1\\ d-odd}}\left\{q_n(d,k)^{(0,1)}_{i_1,n} \mid k = i_1 - d\right\}\right].
\end{aligned}
$$

(c) If $i_1 = i_2 = n$, then

$$
[W_{\Omega_n}\backslash W/W_{\Omega_n}] = \bigcup_{\substack{0\le d\le n\\ d-even}} \{q_n(d,k)^{(0,0)}_{n,n} \mid k = n - d\}.
$$

PROOF. We know that $[W_\Theta\backslash W/W_\Omega] = [W_\Theta\backslash W]\cap[W/W_\Omega]$, for $\Theta, \Omega \subset \Delta$. From Lemmas 5.3 and 5.4, we have

$$
\begin{aligned}
[W_{\Omega_{i_1}}\backslash W/W_{\Omega_{i_2}}] = & \quad [W_{\Omega_{i_1}}\backslash W] \cap [W/W_{\Omega_{i_2}}]\\
= & \left(\bigcup_{0\le j_1\le i_1} X^{i_1}_{j_1}\right) \cap \left(\bigcup_{0\le j_2\le i_2} Y^{i_2}_{j_2}\right)\\
= & \bigcup_{0\le j_1\le i_1}\bigcup_{0\le j_2\le i_2} \left(X^{i_1}_{j_1} \cap Y^{i_2}_{j_2}\right).
\end{aligned}
$$

Now, Lemma 5.5 tells us when $X^{i_1}_{j_1} \cap Y^{i_2}_{j_2}$ is nonempty and gives the proposition.

(a) If $i_1 = n$, $i_2 < n$, then $q_n(d,k)^{(1,1)}_{n,i_2}$ and $q_n(d,k)^{(0,1)}_{n,i_2}$ are not defined and $q_n(d,k)^{(0,0)}_{n,i_2}$ is not defined for d odd. For d even, the inequality $max\{0,(i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d$ becomes $max\{0, i_2 - d\} \le k \le min\{i_1, i_2\} - d$, and its only solution is $k = i_2 - d$.

(b),(c) Analogously. □

In the same way, we get

PROPOSITION 5.8. *Let $i_2 \in \{1, ..., n\}$. Then,*

$$(i)\quad [W_{\bar{\Omega}_n}\backslash W/W_{\Omega_{i_2}}] = \left[\bigcup_{\substack{0\le d\le i_2\\ d-odd}} \left\{q_n(d,k)^{(-1,-1)}_{n,i_2} \mid k = i_2 - d\right\}\right]$$

$$\cup \left[\bigcup_{\substack{0\le d\le i_2\\ d-even}} \left\{q_n(d,k)^{(\pm1,-1)}_{n,i_2} \mid k = i_2 - d\right\}\right].$$

$$(ii)\quad [W_{\bar{\Omega}_n}\backslash W/W_{\bar{\Omega}_n}] = \bigcup_{\substack{0\le d\le n\\ d-even}} \left\{q_n(d,k)^{(-2,-2)}_{n,n} \mid k = n - d\right\}.$$

In particular, for $i_2 = n$, (i) reduces to

$$[W_{\bar{\Omega}_n}\backslash W/W_{\Omega_n}] = \bigcup_{\substack{0\le d\le n\\ d-odd}} \left\{q_n(d,k)^{(-1,-1)}_{n,n} \mid k = n - d\right\}.$$

LEMMA 5.9. *Fix $i_1, i_2 \in \{1, 2, \ldots, n\}$. Suppose that integers d, d' and k, k' satisfy the following conditions:*

$$0 \le d, d' \le min\{i_1, i_2\},$$
$$max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d,$$
$$max\{0, (i_1 + i_2 - n) - d'\} \le k' \le min\{i_1, i_2\} - d'.$$

Then,

(i) $(p_n(d,k)_{i_1,i_2})^{-1} = p_n(d,k)_{i_2,i_1}$.

(ii) $\left(q_n(d,k)^{(0,0)}_{i_1,i_2}\right)^{-1} = q_n(d,k)^{(0,0)}_{i_2,i_1}$, $\left(q_n(d,k)^{(1,1)}_{i_1,i_2}\right)^{-1} = q_n(d,k)^{(1,1)}_{i_2,i_1}$, $\left(q_n(d,k)^{(1,0)}_{i_1,i_2}\right)^{-1} = q_n(d,k)^{(0,1)}_{i_2,i_1}$.

(iii) *Let $w = q_n(d,k)^{(*)}_{i_1,i_2}$, $w' = q_n(d',k')^{(**)}_{i_1,i_2}$, where $(*), (**) \in \{(0,0), (1,1), (1,0), (0,1)\}$. If $w = w'$, then $(*) = (**)$, $d = d'$, $k = k'$.*

PROOF. The proofs of (i) and (ii) are straightforward calculations (also, cf. Lema 4.7 [T1]).

(iii) Write $w = p\epsilon$ and $w' = p'\epsilon'$, where $p = p_n(d,k)_{i_1,i_2}$ and $p' = p_n(d',k')_{i_1,i_2}$. Suppose that $(*) = (**)$, and that d and d' are both odd or both even. If we compare the numbers of -1's which appear in ϵ and ϵ', we get $d = d'$. Therefore, $p_n(d,k)_{i_1,i_2} = p_n(d,k')_{i_1,i_2}$. The definition of $p_n(d,k)_{i_1,i_2}$ implies that k is the maximal integer which satisfies $0 \le k \le min\{i_1, i_2\} - d$ and $p_n(d,k)_{i_1,i_2}(l) = l$ for all $1 \le l \le k$. This implies $k = k'$.

We are now going to prove that in other cases we cannot have $w = w'$.

a) Let $(*) = (0,0)$, d even. Suppose that $(**) = (0,0)$, d' odd. Then $w = w'$ implies $n - i_2 = 0, n - i_2 - 1 = 0$, which is impossible. We can use the same reasoning in the cases $(**) = (1,1)$ and $(**) = (1,0)$. In the case $(**) = (0,1)$, we consider w^{-1} and $(w')^{-1}$. They are of type $(0,0)$ and $(1,0)$, so $w^{-1} \neq (w')^{-1}$, which implies $w \neq w'$.

b) Let $(*) = (0,0)$, d odd. Then $i_1, i_2 < n$. Suppose that $(**) = (1,1)$. Then $w = w'$ implies $d = d' + 1$, $k = k' = i_1 + i_2 - n - d'$. Now we have

$$\begin{aligned} i_1 + i_2 - d - k &= i_1 + i_2 - d - (i_1 + i_2 - n - d + 1) = n - 1, \\ i_1 + i_2 - d' - k &= n, \end{aligned}$$

and by definition

$$p_n(d,k)_{i_1,i_2}(n) = n, \quad p_n(d',k)_{i_1,i_2}(n) = n - i_2 + k = i_1 - d' < n,$$

which contradicts the assumption $w = w'$.

If we suppose $(**) = (1,0)$, then we get $d = d'$, $k = k'$. But the condition for w is $k > i_1 + i_2 - n - d$, and for w' $k = i_1 + i_2 - n - d$. This is again a contradiction.

In the case $(**) = (0,1)$, the equality $w = w'$ implies $n - i_2 = 0$ and $n - i_2 - 1 = 0$, which is impossible.

c) Let $(*) = (1,1)$, d even. Suppose that $(**) = (1,0)$. Then $w = w'$ implies $d' = d + 1$, $k = k'$. But we have $k = i_1 + i_2 - n - d$ and $k' = i_1 + i_2 - n - d' = i_1 + i_2 - n - d - 1$, which is impossible. The assumption $(**) = (0,1)$ gives $n - i_2 - 1 = 0$ and $n - i_2 = 0$, a contradiction.

d) Let $(*) = (1,0)$, $(**) = (0,1)$. Then $w = w'$ implies $n - i_2 - 1 = 0$, $n - i_2 = 0$, which is impossible. □

Define $q_n(d,k)_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2})$. This is an automorphism of Σ. If d is even, then $q_n(d,k)_{i_1,i_2}$ is an element of W. Recall that for $i \in \{1, \ldots, n\}$, we defined

$$\Omega_i = \begin{cases} \Delta \backslash \{\alpha_i\}, & i \neq n-1, \\ \Delta \backslash \{\alpha_n, \alpha_{n-1}\}, & i = n-1, \end{cases}, \quad \bar{\Omega}_n = \Delta \backslash \{\alpha_{n-1}\}, \quad \Omega_0 = \Delta.$$

LEMMA 5.10. *Let $w = q_n(d,k)_{i_1,i_2}$. Then,*

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}.$$

PROOF. The conditions on d and k imply

$$\begin{aligned} 0 &\leq k \leq i_1 - d \leq i_1 \leq i_1 + i_2 - d - k \leq n, \\ 0 &\leq k \leq i_2 - d \leq i_2 \leq i_1 + i_2 - d - k \leq n. \end{aligned}$$

Set

$$\begin{aligned} \beta_i &= \alpha_i, \qquad i = 1, \ldots, n-1, \\ \beta_n &= e_n. \end{aligned}$$

Then $\Gamma = \{\beta_1, \dots, \beta_n\}$ is the set of simple roots of the root system of type B_n, into which our root system embeds. We shall use the following formula, proved in [T1]:

$$(*) \qquad (\Gamma \setminus \{\beta_{i_1}\}) \cap w(\Gamma \setminus \{\beta_{i_2}\}) = \Gamma \setminus \{\beta_l \mid l \in \{k, i_1 - d, i_1, i_1 + i_2 - d - k\} \setminus \{0\}\}.$$

Since we have

$$\Gamma \setminus \{\beta_n\} = \Delta \setminus \{\alpha_n\},$$
$$(\Gamma \setminus \{\beta_j\}) \setminus \{\beta_n\} = \Omega_j \setminus \{\alpha_n\}, \quad (\text{for } j \neq 0),$$

it follows that

$$(**) \qquad (\Omega_{i_1} \cap w(\Omega_{i_2})) \setminus \{\alpha_n, w(\alpha_n)\} = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n, w(\alpha_n)\}.$$

We consider several cases.

(a) Let $i_1, i_2 < n-1$. First, suppose that $i_1 + i_2 - d - k < n - 1$. Then $k, i_1 - d < n - 1$. From the definition of $w = q_n(d,k)_{i_1,i_2}$, we get $w(e_{n-1}) = e_{n-1}$, $w(e_n) = e_n$, so $w(\alpha_{n-1}) = \alpha_{n-1}$, $w(\alpha_n) = \alpha_n$. We apply this to formula (*) and we get

$$(\Delta \setminus \{\alpha_{i_1}\}) \cap w(\Delta \setminus \{\alpha_{i_2}\}) = \Delta \setminus \{\alpha_l \mid l \in \{k, i_1 - d, i_1, i_1 + i_2 - d - k\} \setminus \{0\}\}.$$

Since $i_1, i_2, k, i_1 - d, i_1 + i_2 - d - k < n - 1$, this is exactly the formula from the lemma.

Next, we consider the case when $i_1 + i_2 - d - k = n - 1$. Then,

$$w^{-1}(\alpha_{n-1}) = w^{-1}(e_{n-1} - e_n) = p_n(d,k)_{i_2,i_1}(e_{n-1} - e_n) = e_{i_2-d} - e_n.$$

This is an element of Δ if $i_2 - d = n - 1$, which is impossible since $i_2 < n - 1$. So, $w^{-1}(\alpha_{n-1}) \notin \Delta$ and $\alpha_{n-1} \notin w(\Omega_{i_2})$. In the same way, we see that $\alpha_n \notin w(\Omega_{i_2})$. Hence,

$$\Omega_{i_1} \cap w(\Omega_{i_2}) \subseteq \Delta \setminus \{\alpha_{n-1}, \alpha_n\} = \Omega_{n-1}.$$

Similarly, $w(\alpha_n) \notin \Delta$. Now from (**), we have

$$(\Omega_{i_1} \cap w(\Omega_{i_2})) \setminus \{\alpha_n\} = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n\}.$$

Since $\Omega_{i_1} \cap w(\Omega_{i_2}) \subseteq \Omega_{n-1}$ and $i_1 + i_2 - d - k = n - 1$, if we intersect the above equality with Ω_{n-1}, we get the formula we need.

It remains to consider the case when $i_1 + i_2 - d - k = n$. We have

$$w^{-1}(\alpha_n) = e_{i_2-d-1} + e_{i_2-d},$$

and this is not in Δ because $i_2 < n - 1$. Hence, $\alpha_n \notin w(\Omega_{i_2})$. Also, we have

$$w(\alpha_n) = e_{i_1-d-1} + e_{i_1-d},$$

and again this is not in Δ. Now, the relation (**) becomes

$$\begin{aligned}\Omega_{i_1} \cap w(\Omega_{i_2}) &= (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n\} \\ &= \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k},\end{aligned}$$

since $\Omega_{i_1+i_2-d-k} = \Omega_n = \Delta \setminus \{\alpha_n\}$.

(b) Let $i_1 \geq n-1$. If $i_1+i_2-d-k = n-1$ (this is possible for $i_1 = n-1$), then we have

$$w(\alpha_n) = w(e_{n-1}+e_n) = e_{n-1-i_2+k} + e_n.$$

If $i_1+i_2-d-k = n$, then

$$w(\alpha_n) = w(e_{n-1}+e_n) = e_{n-1-i_2+k} + e_{n-i_2+k}.$$

Anyway, $w(\alpha_n) \in \Delta$ implies $w(\alpha_n) = \alpha_n$, and the relation (**) becomes

$$(\Omega_{i_1} \cap w(\Omega_{i_2})) \setminus \{\alpha_n\} = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n\}.$$

Since $\alpha_n \notin \Omega_{i_1}$, the relation we want follows immediately.

(c) Now, consider the case when $i_1 < n-1$, $i_2 = n-1$. First, suppose that $i_1+i_2-d-k = n-1$. If $d = 0$, then $w(\alpha_n) = w(e_{n-1}+e_n) = e_{n-1}+e_n = \alpha_n$, so $\alpha_n \notin w(\Omega_{i_2})$ and $w(\alpha_n) = \alpha_n \notin \Omega_{i_2}$. If $d > 0$, then

$$\begin{aligned}w^{-1}(\alpha_n) &= w^{-1}(e_{n-1}+e_n) \\ &= p_n(d,k)_{i_2,i_1}(e_{n-1}+e_n) = e_{i_2-d} + e_n \notin \Delta, \\ w(\alpha_n) &= w(e_{n-1}+e_n) = p_n(d,k)_{i_1,i_2}(-e_{n-1}+e_n) \\ &= -e_{i_1-d+1} + e_n \notin \Delta.\end{aligned}$$

Anyway, the relation (**) becomes

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n\},$$

and since $i_1+i_2-d-k = n-1$, we are done. Now, suppose that $i_1+i_2-d-k = n$. Then

$$w^{-1}(\alpha_n) = p_n(d,k)_{i_2,i_1}(e_{n-1}+e_n) = e_{i_2-d-1} + e_{i_2-d},$$

and this is not in Δ since $i_2 - d \leq n-1$. If $d = 0$, then

$$w(\alpha_n) = w(e_{n-1}+e_n) = p_n(d,k)_{i_1,i_2}(e_{n-1}+e_n) = e_n + e_{i_1} \notin \Delta,$$

and the relation (**) gives the result.

If $d > 0$, then

$$\begin{aligned}&w(\alpha_n) = w(e_{n-1}+e_n) = p_n(d,k)_{i_1,i_2}(-e_{n-1}+e_n) \\ &= -e_{i_1-d+1} + e_{i_1-d} = \alpha_{i_1-d}.\end{aligned}$$

Now from (**), we have

$$(\Omega_{i_1} \cap w(\Omega_{i_2})) \setminus \{\alpha_{i_1-d}\} = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n, \alpha_{i_1-d}\}.$$

But $\alpha_{i_1-d} \notin w(\Omega_{i_2})$ since $\alpha_{i_1-d} = w(\alpha_n)$. Also, $\alpha_n \notin \Omega_{i_1+i_2-d-k}$ and $\alpha_{i_1-d} \notin \Omega_{i_1-d}$. The result follows.

(d) It remains to consider the case $i_1 < n-1$, $i_2 = n$. Then $i_1+i_2-d-k = n$ and we have

$$w^{-1}(\alpha_n) = p_n(d,k)_{i_2,i_1}(e_{n-1}+e_n) = e_{n-1-i_1+k} + e_{n-i_1+k} = e_{n-1-d} + e_{n-d}.$$

We see that $w^{-1}(\alpha_n) \notin \Delta$, for $d > 0$, and $w^{-1}(\alpha_n) = \alpha_n$, for $d = 0$. Since $i_2 = n$, in both cases we have $\alpha_n \notin w(\Omega_{i_2})$. Now for $d > 1$, we have

$$w(\alpha_n) = p_n(d,k)_{i_1,i_2}(-e_{n-1} - e_n),$$

and for $d = 1$

$$w(\alpha_n) = p_n(d,k)_{i_1,i_2}(e_{n-1} - e_n) = e_n - e_{i_1},$$

and in both cases $w(\alpha_n) \notin \Delta$. Hence, relation (**) becomes

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = (\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}) \setminus \{\alpha_n\},$$

and the result follows from the condition $i_1 + i_2 - d - k = n$. □

LEMMA 5.11. *(i) If $w = q_n(d,k)^{(0,0)}_{i_1,i_2}$ or $w = q_n(d,k)^{(1,0)}_{i_1,i_2}$, then*

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \bigcap_{j\in\Pi} \Omega_j \quad \textit{where } \Pi = \{k, i_1-d, i_1, i_1+i_2-d-k\}\setminus\{0\},$$

or, equivalently,

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_{i_1+i_2-d-k}.$$

(ii) If $w = q_n(d,k)^{(1,1)}_{i_1,i_2}$ or $w = q_n(d,k)^{(0,1)}_{i_1,i_2}$, then

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \bar{\Omega}_n,$$

or, equivalently,

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = s\left(\bigcap_{j\in\Pi} \Omega_j\right) \quad \Pi = \{k, i_1-d, i_1, i_1+i_2-d-k=n\},$$

where $s = (\mathbf{1}_{n-1}, -1)$ denotes the automorphism of Σ which interchanges α_{n-1} and α_n.

PROOF. (a) Let $w = q_n(d,k)^{(0,0)}_{i_1,i_2}$. If d is even, then $w = q_n(d,k)_{i_1,i_2}$, and the statement follows from Lemma 5.10. If d is odd, then $i_1, i_2 < n$. Now $w = w's$, where $w' = q_n(d,k)_{i_1,i_2}$. Note that $s(\Omega_{i_2}) = \Omega_{i_2}$ for $i_2 < n$, so we have

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \Omega_{i_1} \cap w's(\Omega_{i_2}) = \Omega_{i_1} \cap w'(\Omega_{i_2}).$$

The result follows from Lemma 5.10.

(b) Let $w = q_n(d,k)^{(1,1)}_{i_1,i_2}$. Then, $i_1, i_2 < n$, $i_1+i_2-d-k = n$ and $w = sw's$, where $w' = q_n(d,k)_{i_1,i_2}$. Now, we have

$$\begin{aligned}\Omega_{i_1} \cap w(\Omega_{i_2}) &= \Omega_{i_1} \cap sw's(\Omega_{i_2}) = s(\Omega_{i_1} \cap w's(\Omega_{i_2})) = \\ &= s(\Omega_{i_1} \cap w'(\Omega_{i_2})) = (\text{Lemma } 5.10) \\ &= s(\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_n) = \\ &= \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \bar{\Omega}_n.\end{aligned}$$

(c) Let $w = q_n(d,k)^{(1,0)}_{i_1,i_2}$. Then, $i_2 < n$ and $w = w's$, where $w' = q_n(d,k)_{i_1,i_2}$. It follows that

$$\Omega_{i_1} \cap w(\Omega_{i_2}) = \Omega_{i_1} \cap w's(\Omega_{i_2}) = \Omega_{i_1} \cap w'(\Omega_{i_2})$$

and Lemma 5.10 gives the result.

(d) Let $w = q_n(d,k)^{(0,1)}_{i_1,i_2}$. Then, $i_1 < n$ and $w = sw'$, where $w' = q_n(d,k)_{i_1,i_2}$. Now, we have

$$\begin{aligned}\Omega_{i_1} \cap w(\Omega_{i_2}) &= \Omega_{i_1} \cap sw'(\Omega_{i_2}) = s(\Omega_{i_1} \cap w'(\Omega_{i_2})) = \\ &= (\text{Lemma } 5.10) = s(\Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \Omega_n) = \\ &= \Omega_k \cap \Omega_{i_1-d} \cap \Omega_{i_1} \cap \bar{\Omega}_n.\end{aligned}$$

□

In the same way, we get

LEMMA 5.12. *Let $w = q_n(d,k)^{(-1,-1)}_{n,i_2}$ or $w = q_n(d,k)^{(\pm 1,-1)}_{n,i_2}$. Then,*

$$\bar{\Omega}_n \cap w(\Omega_{i_2}) = s(\Omega_k \cap \Omega_{n-d} \cap \Omega_n).$$

6. ORTHOGONAL GROUP $O(2n,F)$

The orthogonal group $O(2n,F)$, $n \geq 1$, is the group

$$O(2n,F) = \{X \in GL(2n,F) |\ {}^{\tau}XX = I_{2n}\}.$$

$O(2n,F)$ has two connected components. The first is $SO(2n,F) = \{X \in O(2n,F) \mid detX = 1\}$, and the second is $\{X \in O(2n,F) \mid detX = -1\}$. We have

$$O(2n,F) = SO(2n,F) \cup s \cdot SO(2n,F),$$

where

$$s = \begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{bmatrix}.$$

Let Δ denote the set of simple roots of $SO(2n,F)$, W the Weyl group. Let $\alpha = (n_1, ..., n_k)$ be an ordered partition of $m \leq n$. Denote by $P_\alpha = M_\alpha U_\alpha$ the standard parabolic subgroup of $SO(2n,F)$ with Levi factor $M_\alpha \cong GL(n_1,F) \times$

$\cdots \times GL(n_k, F) \times SO(2(n-m), F)$. We shall consider the following subgroups of $O(2n, F)$:

$$Q_\alpha = \begin{cases} P_\alpha \cup sP_\alpha, & \text{for } m < n, \\ P_\alpha, & \text{for } m = n. \end{cases}$$

It follows that $Q_\alpha = N_\alpha U_\alpha$, where

$$N_\alpha = \begin{cases} M_\alpha \cup sM_\alpha, & \text{for } m < n, \\ M_\alpha, & \text{for } m = n. \end{cases}$$

We have

$$N_\alpha = \{diag(g_1, ..., g_k, h, \ ^\tau g_k^{-1}, ..., \ ^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), \\ h \in O(2(n-m), F)\},$$

so

$$N_\alpha \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times O(2(n-m), F).$$

Let $\alpha = (i)$. The subgroups $N = N_\alpha$ and $V = U_\alpha$ are closed, N normalises V and $N \cap V = \{e\}$, so by the first section, we have functors $i_{V,1}$ and $r_{V,1}$. Define $i_{G,N} = i_{V,1}$ and $r_{N,G} = r_{V,1}$. Hence,

$$i_{G,N} : AlgN \to AlgG, \ r_{N,G} : AlgG \to AlgN.$$

Let $\alpha = (i_1)$, $\beta = (i_2)$, $P = Q_\alpha = MU$, $Q = Q_\beta = NV$.Let σ be an admissible representation of $O(2n, F)$. We consider

$$r_{N,G} \circ i_{G,M}(\sigma).$$

By Theorem 2.1, we can find a composition series of $r_{N,G} \circ i_{G,M}(\sigma)$. We need to calculate representatives of

$$P \setminus O(2n, F) \ / \ Q.$$

LEMMA 6.1. *Let* $i_1, i_2 \in \{1, ..., n\}$, $\alpha = (i_1)$, $\beta = (i_2)$, $P = Q_\alpha = MU$, $Q = Q_\beta = NV$.

(i) $\{q_n(d,k)_{i_1,i_2} \mid 0 \le d \le min\{i_1, i_2\}, max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d\}$ *is a set of representatives of* $P \backslash O(2n, F) \ / \ Q$.

(ii) *Let* $w = q_n(d,k)_{i_1,i_2}$. *The groups* $w^{-1}(P)$, $w^{-1}(M)$ *and* $w^{-1}(U)$ *are decomposable with respect to* (N, V), *and the groups* $w(Q)$, $w(N)$ *and* $w(V)$ *are decomposable with respect to* (M, U).

PROOF. (a) Suppose that $i_1, i_2 < n$. Then,

$$P = Q_\alpha = P_\alpha \cup sP_\alpha, \quad Q = Q_\beta = P_\beta \cup sP_\beta.$$

Let $x \in SO(2n, F)$. Then,

$$PxsQ = (P_\alpha \cup sP_\alpha)\ x\ (sP_\beta \cup P_\beta) = PxQ,$$

so $[x] = [xs]$. Analogously, $[x] = [sx]$. Thus, we can choose representatives from $SO(2n, F)$. Let $x, y \in SO(2n, F)$ with $[x] = [y]$. Now, we have

$$\begin{aligned} PxQ &= PyQ, \\ (P_\alpha \cup sP_\alpha)\ x\ (sP_\beta \cup P_\beta) &= (P_\alpha \cup sP_\alpha)\ y\ (sP_\beta \cup P_\beta), \end{aligned}$$

$$(P_\alpha x P_\beta) \cup (sP_\alpha x P_\beta) \cup (P_\alpha x s P_\beta) \cup (P_\alpha s x s^{-1} P_\beta) =$$

$$(P_\alpha y P_\beta) \cup (sP_\alpha y P_\beta) \cup (P_\alpha y s P_\beta) \cup (P_\alpha s y s^{-1} P_\beta).$$

It follows that

$$(P_\alpha x P_\beta) \cup (P_\alpha s x s^{-1} P_\beta) = (P_\alpha y P_\beta) \cup (P_\alpha s y s^{-1} P_\beta),$$

so

$$\begin{aligned} &P_\alpha x P_\beta = P_\alpha y P_\beta \qquad \text{or} \\ &P_\alpha x P_\beta = P_\alpha s y s^{-1} P_\beta. \end{aligned}$$

We know that $[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ is a set of representatives of $P_\alpha \setminus SO(2n, F)$ / P_β. By the above considerations, a set of representatives of $P \setminus O(2n, F) / Q$ can be chosen from the set $[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ in the following way: we take all elements which satisfy $P_\alpha w P_\beta = P_\alpha s w s^{-1} P_\beta$, and from the remaining set we choose w or a representative of $P_\alpha s w s^{-1} P_\beta$. For $w = q_n(d, k)_{i_1, i_2}^{(0,0)} \in [W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ and $i_1 + i_2 - d - k < n$, we have $w = sws^{-1}$. For $i_1 + i_2 - d - k = n$, we have

$$\begin{aligned} s q_n(d,k)_{i_1,i_2}^{(0,0)} s^{-1} &= q_n(d,k)_{i_1,i_2}^{(1,1)}, \quad \text{if } d \text{ is even}, \\ s q_n(d,k)_{i_1,i_2}^{(1,0)} s^{-1} &= q_n(d,k)_{i_1,i_2}^{(0,1)}, \quad \text{if } d \text{ is odd}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\{q_n(d,k)_{i_1,i_2}^{(0,0)} \mid 0 \le d \le min\{i_1, i_2\}, \ \text{for } d \text{ even} \\ &\qquad\qquad max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d, \\ &\qquad\qquad max\{0, (i_1 + i_2 - n) - d\} < k \le min\{i_1, i_2\} - d \text{ for } d \text{ odd}\} \\ &\cup\{q_n(d,k)_{i_1,i_2}^{(1,0)} \mid 0 \le d \le min\{i_1, i_2\}, \quad d \text{ odd}, \ k = i_1 + i_2 - n - d \ge 0\} \end{aligned}$$

is a set of representatives of $P \setminus O(2n, F) / Q$. We have

$$q_n(d,k)_{i_1,i_2} = \begin{cases} q_n(d,k)_{i_1,i_2}^{(0,0)}, & \text{for } d \text{ even}, \\ q_n(d,k)_{i_1,i_2}^{(0,0)} s, & \text{for } d \text{ odd and } i_1 + i_2 - d - k < n, \\ q_n(d,k)_{i_1,i_2}^{(1,0)} s, & \text{for } d \text{ odd and } i_1 + i_2 - d - k = n. \end{cases}$$

Then, from the relation $[x] = [sx]$, it follows that the set

$$\{q_n(d,k)_{i_1,i_2} \mid 0 \le d \le min\{i_1, i_2\},\ max\{0, (i_1+i_2-n)-d\} \le k \le \\ \le min\{i_1,i_2\} - d\}$$

is a set of representatives of $P \setminus O(2n, F) \;/\; Q$.

Let $w = q_n(d,k)_{i_1,i_2}$. We shall show that the group $w(Q)$ is decomposable with respect to (M, U).

If $i_1 + i_2 - d - k < n$, then w and s commute, so

$$w(s) = wsw^{-1} = s.$$

Then,

$$\begin{aligned} w(Q) \cap (MU) &= wQw^{-1} \cap (MU) \\ &= w(P_\beta \cup sP_\beta) \cap (M_\alpha U_\alpha \cup sM_\alpha U_\alpha) \\ &= [w(P_\beta) \cap M_\alpha U_\alpha] \cup s\,[w(P_\beta) \cap M_\alpha U_\alpha] = \end{aligned}$$

(because $w(P_\beta)$ is decomposable with respect to (M_α, U_α))

$$\begin{aligned} &= [(w(P_\beta) \cap M_\alpha)(w(P_\beta) \cap U_\alpha)] \\ &\quad \cup s\,[(w(P_\beta) \cap M_\alpha)(w(P_\beta) \cap U_\alpha)] \\ &= [(w(P_\beta) \cap M_\alpha) \cup s(w(P_\beta) \cap M_\alpha)]\,[w(P_\beta) \cap U_\alpha]\,. \end{aligned}$$

On the other side,

$$\begin{aligned} &(w(Q) \cap M)(w(Q) \cap U) = \\ &\quad [(w(P_\beta) \cup s \cdot w(P_\beta)) \cap (M_\alpha \cup sM_\alpha)]\,[(w(P_\beta) \cup s \cdot w(P_\beta)) \cap U_\alpha] = \\ &\qquad [(w(P_\beta) \cap M_\alpha) \cup s \cdot (w(P_\beta) \cap M_\alpha)]\,[w(P_\beta) \cap U_\alpha]\,, \end{aligned}$$

so $w(Q)$ is decomposable with respect to (M, U).

If $i_1 + i_2 - d - k = n$, then

$$w(Q) \cap P = w(P_\beta \cup sP_\beta) \cap (P_\alpha \cup sP_\alpha) = (w(P_\beta) \cap P_\alpha) \cup (w(sP_\beta) \cap sP_\alpha).$$

It can be shown that

$$w(sP_\beta) \cap sP_\alpha = \emptyset,$$

which implies $w(sP_\beta) \cap sM_\alpha = \emptyset$. It follows that

$$\begin{aligned} w(Q) \cap P &= w(P_\beta) \cap P_\alpha, \\ w(Q) \cap M &= w(P_\beta) \cap M_\alpha. \end{aligned}$$

If $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, we have

$$\begin{aligned} w(Q) \cap MU &= w(P_\beta) \cap P_\alpha = w(P_\beta) \cap M_\alpha U_\alpha = \\ &\quad \text{(since } w(P_\beta) \text{ is decomposable with respect to } (M_\alpha, U_\alpha)) \\ &= (w(P_\beta) \cap M_\alpha)(w(P_\beta) \cap U_\alpha) = (w(Q) \cap M)(w(Q) \cap U). \end{aligned}$$

If $w \notin \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$,then $w' = ws \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ and

$$w'(P_\beta) = w' P_\beta (w')^{-1} = w s P_\beta s w^{-1} = w P_\beta w^{-1}.$$

Now, we have

$$\begin{aligned} w(Q) \cap MU &= w(P_\beta) \cap P_\alpha = w'(P_\beta) \cap M_\alpha U_\alpha = \\ &= (w'(P_\beta) \cap M_\alpha)(w'(P_\beta) \cap U_\alpha) = (w(Q) \cap M)(w(Q) \cap U). \end{aligned}$$

Hence, $w(Q)$ is decomposable with respect to (M, U).

For the other groups, the proof is similar.

(b) Let $i_1 = n, \quad i_2 < n$. For $x \in SO(2n, F)$, we have

$$PxsQ = P_\alpha \; x \; (sP_\beta \cup P_\beta) = PxQ,$$

so $[x] = [xs]$ (but the classes $[x]$ and $[sx]$ are not the same in general). Hence, we can choose representatives from $SO(2n, F)$ again. Let $x, y \in SO(2n, F)$ and $[x] = [y]$. Now,

$$\begin{aligned} P_\alpha \; x \; (sP_\beta \cup P_\beta) &= P_\alpha \; y \; (sP_\beta \cup P_\beta), \\ (P_\alpha x P_\beta) \cup (P_\alpha x s P_\beta) &= (P_\alpha y P_\beta) \cup (P_\alpha y s P_\beta). \end{aligned}$$

It follows that

$$P_\alpha x P_\beta = P_\alpha y P_\beta,$$

so $\left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ is a set of representatives of $P \setminus O(2n, F) \;/\; Q$. Recall that

$$\begin{aligned} \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right] &= \left[\bigcup_{\substack{0 \le d \le i_2 \\ d-even}} \left\{ q_n(d,k)^{(0,0)}_{n,i_2} \mid k = i_2 - d \right\}\right] \\ &\cup \left[\bigcup_{\substack{0 \le d \le i_2 \\ d-odd}} \left\{ q_n(d,k)^{(1,0)}_{n,i_2} \mid k = i_2 - d \right\}\right] \end{aligned}$$

Since

$$q_n(d,k)_{n,i_2} = \begin{cases} q_n(d,k)^{(0,0)}_{n,i_2}, & \text{for } d \text{ even}, \\ q_n(d,k)^{(1,0)}_{n,i_2} s, & \text{for } d \text{ odd}, \end{cases}$$

the equality $[x] = [xs]$ implies that

$$\begin{aligned} &\{q_n(d,k)_{n,i_2} \mid 0 \le d \le min\{i_1, i_2\}, \\ &\quad max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d\} \end{aligned}$$

is a set of representatives of $P \setminus O(2n, F) \;/\; Q$.

Let $w = q_n(d,k)_{n,i_2}$. Then, $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ or $ws \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$. Since $i_2 < n$, we have $(ws)(P_\beta) = w(P_\beta)$. Now,

$$\begin{aligned} w(Q) \cap MU &= w(P_\beta \cup sP_\beta) \cap M_\alpha U_\alpha = w(P_\beta) \cap M_\alpha U_\alpha = \\ &= (w(P_\beta) \cap M_\alpha)(w(P_\beta) \cap U_\alpha) = (w(Q) \cap M)(w(Q) \cap U). \end{aligned}$$

The proof for $w(N)$ and $w(V)$ is similar. If $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, then it is easy to show the statement for $w^{-1}(P)$. If $ws \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, then

$$\begin{aligned} w^{-1}(P) \cap NV &= w^{-1}(P_\alpha) \cap (P_\beta \cup sP_\beta) = w^{-1}(P_\alpha) \cap (P_\beta) \\ &= s \cdot (sw^{-1}(P_\alpha) \cap (P_\beta)) \cdot s \end{aligned}$$

(since $(sw^{-1})(P_\alpha)$ is decomposable with respect to (M_β, U_β))

$$\begin{aligned} &= s \cdot ((sw^{-1})(P_\alpha) \cap M_\beta)((sw^{-1})(P_\alpha) \cap U_\beta) \cdot s \\ &= (w^{-1}(P_\alpha) \cap M_\beta)(w^{-1}(P_\alpha) \cap U_\beta) \\ &= (w^{-1}(P_\alpha) \cap N)(w^{-1}(P_\alpha) \cap V). \end{aligned}$$

Analogously for $w^{-1}(M)$, $w^{-1}(U)$.

(c) For $i_1 < n$, $i_2 = n$, the argument is analogous to (b).

(d) Let $i_1 = i_2 = n$. For $x, y \in SO(2n, F)$, we have

$$[x] = [y] \Leftrightarrow P_\alpha x P_\beta = P_\alpha y P_\beta,$$

so elements of $\left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ represent different classes. Moreover,

$$\begin{aligned} P_\alpha x P_\beta &\subseteq SO(2n, F), \\ P_\alpha s x P_\beta &\subseteq O(2n, F) \backslash SO(2n, F), \end{aligned}$$

so

$$[x] \neq [sx], \qquad [sx] \neq [y].$$

Let $x, y \in SO(2n, F)$, $[sx] = [sy]$. Then,

$$\begin{aligned} P_\alpha s x P_\beta &= P_\alpha s y P_\beta, \\ sP_\alpha s x P_\beta &= sP_\alpha s y P_\beta. \end{aligned}$$

We conclude that the elements sw, where $w \in \left[W_{\bar{\Omega}_n} \backslash W / W_{\Omega_n}\right]$, represent all the classes of type $[sx]$, $x \in SO(2n, F)$. Now, we get the following set of representatives:

$$\begin{aligned} &\left[\bigcup_{\substack{0 \le d \le n \\ d-even}} \left\{ q_n(d,k)_{n,n}^{(0,0)} \mid k = n - d \right\} \right] \\ \cup &\left[\bigcup_{\substack{0 \le d \le n \\ d-odd}} \left\{ sq_n(d,k)_{n,n}^{(-1,-1)} \mid k = n - d \right\} \right]. \end{aligned}$$

Since

$$q_n(d,k)_{n,n} = \begin{cases} q_n(d,k)_{n,n}^{(0,0)}, & \text{for } d \text{ even}, \\ sq_n(d,k)_{n,n}^{(-1,-1)}, & \text{for } d \text{ odd}, \end{cases}$$

it follows that

$$\{q_n(d,k)_{n,n} \mid 0 \le d \le min\{i_1,i_2\},$$
$$max\{0,(i_1+i_2-n)-d\} \le k \le min\{i_1,i_2\}-d\}$$

is a set of representatives of $P \setminus O(2n,F) \,/\, Q$.

For $w \in [W_{\Omega_n} \backslash W / W_{\Omega_n}]$, we know by [BZ2] that $w^{-1}(P)$, $w^{-1}(M)$ and $w^{-1}(U)$ are decomposable with respect to (N,V), and that $w(Q)$, $w(N)$ and $w(V)$ are decomposable with respect to (M,U).

Let $w \in \left[W_{\bar{\Omega}_n} \backslash W / W_{\Omega_n}\right]$. Then,

$$\begin{aligned}(sw)^{-1}(P) \cap NV &= (sw)^{-1}(P_\alpha) \cap M_\beta U_\beta \\ &= w^{-1} s P_\alpha s w \cap M_\beta U_\beta =\end{aligned}$$

(since $w^{-1}(sP_\alpha s)$ is decomposable with respect to (M_β, U_β))

$$\begin{aligned}&= (w^{-1} s P_\alpha s w \cap M_\beta)(w^{-1} s P_\alpha s w \cap U_\beta) \\ &= \left[(sw)^{-1}(P) \cap N\right]\left[(sw)^{-1}(P) \cap V\right].\end{aligned}$$

The arguments for $(sw)^{-1}(M)$ and $(sw)^{-1}(U)$ are similar. For $(sw)(Q)$, we have

$$(sw)(Q) \cap MU = (sw)(P_\beta) \cap M_\alpha U_\alpha = s \cdot (w(P_\beta) \cap sM_\alpha s s U_\alpha s) \cdot s =$$

(since $w(P_\beta)$ is decomposable with respect to $(sM_\alpha s, sU_\alpha s)$)

$$\begin{aligned}&= s \cdot (w(P_\beta) \cap sM_\alpha s)(w(P_\beta) \cap sU_\alpha s) \cdot s \\ &= [(sw)(P_\beta) \cap M_\alpha]\,[(sw)(P_\beta) \cap U_\alpha] \\ &= [(sw)(Q) \cap M]\,[(sw)(Q) \cap U].\end{aligned}$$

The arguments for $(sw)(N)$ and $(sw)(V)$ are similar. □

It can be easily verified that, in our case, the character ε from Theorem 2.1 is equal to 1. Now, by Theorem 2.1 and Lemma 6.1, we have

LEMMA 6.2. *Let $i_1, i_2 \in \{1, ..., n\}$, $\alpha = (i_1)$, $\beta = (i_2)$, $P = Q_\alpha = MU$, $Q = Q_\beta = NV$. Let σ be an admissible representation of M. Then $r_{N,G} \circ i_{G,M}(\sigma)$ has a composition series with factors*

$$i_{N,N'} \circ w^{-1} \circ r_{M',M}(\sigma),$$

where $N' = w^{-1}(M) \cap N$, $M' = M \cap w(N)$ and $w \in \{q_n(d,k)_{i_1,i_2} \mid 0 \le d \le min\{i_1,i_2\}, max\{0,(i_1+i_2-n)-d\} \le k \le min\{i_1,i_2\}-d\}$.

The following lemma describes M' and N' from Lemma 6.2.

LEMMA 6.3. *Let* $w = q_n(d,k)_{i_1,i_2}$, $\alpha = (i_1)$, $\beta = (i_2)$. *Then,*

$$N_\alpha \cap w(N_\beta) = N_\gamma,$$

where $\gamma = (k, i_1 - d, i_1, i_1 + i_2 - d - k)$.

PROOF. Recall from [C] (Proposition 1.3.3) that for θ, $\Omega \subseteq \Delta$ and $w \in [W_\theta \backslash W / W_\Omega]$ we have $M_\theta \cap w(M_\Omega) = M_{\theta \cap w(\Omega)}$.

a) Let $i_1, i_2 < n$. If d is even, then $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ and we have

$$\begin{aligned} N_\alpha \cap w(N_\beta) &= (M_\alpha \cup sM_\alpha) \cap w(M_\beta \cup sM_\beta) \\ &= (M_\alpha \cap w(M_\beta)) \cup (sM_\alpha \cap w(sM_\beta)). \end{aligned}$$

If $i_1 + i_2 - d - k < n$, then s and w commute, so

$$N_\alpha \cap w(N_\beta) = (M_\alpha \cap w(M_\beta)) \cup s \cdot (M_\alpha \cap w(M_\beta)) = M_\gamma \cup sM_\gamma = N_\gamma.$$

If $i_1 + i_2 - d - k = n$, then by the proof of Lemma 6.1 we have $sM_\alpha \cap w(sM_\beta) = \emptyset$, so

$$N_\alpha \cap w(N_\beta) = M_\gamma.$$

Since $\gamma = (k, i_1 - d, i_1, i_1 + i_2 - d - k = n)$, it follows that $M_\gamma = N_\gamma$. If d is odd, then $w = w's$, where $w' \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, $w' = q_n(d,k)^{(0,0)}_{i_1,i_2}$, for $i_1 + i_2 - d - k < n$, and $w' = q_n(d,k)^{(1,0)}_{i_1,i_2}$, for $i_1 + i_2 - d - k = n$. Since $i_2 < n$, we have $w(M_\beta) = w's(M_\beta) = w'(M_\beta)$, so the argument is the same.

b) Let $i_1 = n$, $i_2 < n$. If d is even, then $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, so

$$N_\alpha \cap w(N_\beta) = M_\alpha \cap w(M_\beta \cup sM_\beta) = M_\alpha \cap w(M_\beta) = M_\gamma.$$

Since $\gamma = (k, n - d, n)$, we have $M_\gamma = N_\gamma$. If d is odd, then the proof is same, since $w = w's$, where $w' \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$ and $w(M_\beta) = w's(M_\beta) = w'(M_\beta)$.

c) Let $i_1 < n$, $i_2 = n$. If d is even, then $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, and

$$N_\alpha \cap w(N_\beta) = (M_\alpha \cup sM_\alpha) \cap w(M_\beta) = M_\alpha \cap w(M_\beta) = M_\gamma = N_\gamma,$$

because $i_1 + i_2 - d - k = n$. If d is odd, then $w = sw'$, $w' = q_n(d,k)^{(0,1)}_{i_1,i_2}$. Now, we have

$$\begin{aligned} N_\alpha \cap w(N_\beta) &= M_\alpha \cap w(M_\beta) = M_{\Omega_{i_1}} \cap (sw')(M_{\Omega_{i_2}}) \\ &= s(s(M_{\Omega_{i_1}}) \cap w'(M_{\Omega_{i_2}})) = s(M_{\Omega_{i_1}} \cap w'(M_{\Omega_{i_2}})) \\ &= s(M_{\Omega_{i_1} \cap w'(\Omega_{i_2})}) = s(M_{\Omega_k \cap \Omega_{i_1 - d} \cap \Omega_{i_1} \cap \bar{\Omega}_n}) \\ &= M_{\Omega_k \cap \Omega_{i_1 - d} \cap \Omega_{i_1} \cap \Omega_n} = M_\gamma = N_\gamma. \end{aligned}$$

d) Let $i_1 = i_2 = n$. If d is even, then $w \in \left[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}\right]$, and

$$N_\alpha \cap w(N_\beta) = M_\alpha \cap w(M_\beta) = M_\gamma = N_\gamma.$$

If d is odd, then $w = sw'$, $w' = q_n(d,k)^{(-1,-1)}_{i_1,i_2}$. Now, we have

$$\begin{aligned} N_\alpha \cap w(N_\beta) &= M_\alpha \cap w(M_\beta) = s(s(M_\alpha) \cap w'(M_\beta)) = \\ &= s(M_{\bar{\Omega}_n} \cap w'(M_{\Omega_n})) = s(M_{\Omega_k \cap \Omega_{n-d} \cap \bar{\Omega}_n}) = \\ &= M_{\Omega_k \cap \Omega_{n-d} \cap \Omega_n} = N_\gamma. \end{aligned}$$

□

We now do the same construction for even orthogonal groups that we did for $SO(2n,F)$. Let σ be an admissible representation of $O(2n,F)$, π an admissible representation of $GL(m,F)$. Then, $\pi \otimes \sigma$ is a representation of $N_{(m)} \cong GL(m,F) \times O(2n,F)$. Set

$$\pi \rtimes \sigma = i_{N_{(m)},G}(\pi \otimes \sigma),$$

where $G = O(2(m+n),F)$. Note that here we use the notation we introduced at the beginning of this section, so $i_{N_{(m)},G}$ would be $i_{V_{(m)},1}$ if we used the notation from[BZ2].

PROPOSITION 6.4. *Let π, π_1 and π_2 be admissible representations of the groups $GL(n,F)$, $GL(n_1,F)$ and $GL(n_2,F)$, respectively. Let σ be an admissible representation of $O(2m,F)$. Then, $\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$ and $(\pi \rtimes \sigma)^\sim \cong \tilde{\pi} \rtimes \tilde{\sigma}$.*

PROOF. The proof is same as in the case of $SO(2m,F)$, but here we use Proposition 1.9. from [BZ2]. □

Let

$$R(O) = \bigoplus_{n \geq 0} R_n(O),$$

where $R_n(O)$ denotes the Grothendieck group of the category of all finite length smooth representations of $O(2n,F)$. We shall define the structure of an R-module on $R(O)$. First, for irreducible smooth representations $\pi \in R$ and $\sigma \in R(O)$, we put

$$\pi \rtimes \sigma = s.s.(\pi \rtimes \sigma).$$

Now, we extend $\rtimes$ $\mathbb{Z}$-bilinearly to $R \times R(O)$. The action $\rtimes$ induces a $\mathbb{Z}$-linear mapping $\mu : R \otimes R(O) \to R(O)$, which satisfies $\mu(\pi \otimes \sigma) = s.s.(\pi \rtimes \sigma)$ for $\pi \in R$, $\sigma \in R(O)$.

An argument analogous to that for $R(S)$ gives

PROPOSITION 6.5. *$(R(O),\mu)$ is a $\mathbb{Z}_+$-graded module over R.*

We can also achieve an R-comodule structure on $R(O)$. For that purpose, we shall use the Jacquet module. Let σ be a smooth finite length representation of $O(2n,F)$. At the beginning of this section, we defined subgroups Q_α

of $O(2n, F)$, where $\alpha = (n_1, ..., n_k)$ is an ordered partition of a non-negative integer $m \leq n$. We have $Q_\alpha = N_\alpha U_\alpha$, where

$$N_\alpha \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times O(2(n-m), F).$$

Define

$$s_{\alpha,(0)}(\sigma) = r_{N_\alpha, O(2n,F)}(\sigma).$$

(Again, this is the notation from the beginning of this section.) $s_{\alpha,(0)}(\sigma)$ is a representation of N_α, so we may consider $s.s.(s_{\alpha,(0)}(\sigma)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(O)$. For an irreducible smooth representation $\sigma \in R(O)$, we define

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(s_{(k),(0)}(\sigma)).$$

We have $\mu^*(\sigma) \in R \otimes R(O)$. Now, we extend μ^* $\mathbb{Z}$-linearly to $\mu^* : R(O) \to R \otimes R(O)$.

PROPOSITION 6.6. *$(R(O), \mu^*)$ is a $\mathbb{Z}_+$-graded comodule over R.*

7. JACQUET MODULES OF INDUCED REPRESENTATIONS FOR $O(2n, F)$

Lemma 6.2 is the geometric lemma for $O(2n, F)$. If we compare it with the calculations Tadić made in [T1] for $Sp(n, F)$, we see that the geometric lemma is exactly the same for those two groups. Now, we can use the further calculations from [T1] to obtain the formula for $\mu^*(\pi \rtimes \sigma)$.

Let us fix a positive integer n and take $i_1 \in \{1, ..., n\}$. Let π be an admissible representation of $GL(i_1, F)$ and σ an admissible representation of $O(2(n - i_1), F)$.

For $i_2 \in \{1, ..., n\}$, let d and k be an integers which satisfy $0 \leq d \leq min\{i_1, i_2\}$, $max\{0, (i_1 + i_2 - n) - d\} \leq k \leq min\{i_1, i_2\}$. For $w = q_n(d, k)_{i_1, i_2}$, we have

$$w(diag(g_1, g_2, g_3, g_4, h, \,{}^{\tau}g_4^{-1}, \,{}^{\tau}g_3^{-1}, \,{}^{\tau}g_2^{-1}, \,{}^{\tau}g_1^{-1}))w^{-1} =$$

$$= diag(g_1, g_4, \,{}^{\tau}g_3^{-1}, g_2, h, \,{}^{\tau}g_2^{-1}, g_3, \,{}^{\tau}g_4^{-1}, \,{}^{\tau}g_1^{-1}),$$

where $g_1 \in GL(k, F)$, $g_2 \in GL(i_2 - d - k, F), g_3 \in GL(d, F)$, $g_4 \in GL(i_1 - d - k, F)$ and $h \in O(2(n - i_1 - i_2 + d + k), F)$.

LEMMA 7.1. *Let*

$$s.s.(r_{(k,i_1-d-k,d)(i_1)}(\pi)) = \sum_i \pi_i^{(1)} \otimes \pi_i^{(2)} \otimes \pi_i^{(3)},$$

$$s.s.(s_{(i_2-d-k)(0)}(\sigma)) = \sum_j \pi_j^{(4)} \otimes \sigma_j.$$

Let $P = Q_{(i_1)} = MU$, $Q = Q_{(i_2)} = NV$ and $w = q_n(d,k)_{i_1,i_2}$. Then,

$$\begin{aligned} s.s.(i_{N,w^{-1}(M)\cap N} \circ w^{-1} \circ r_{M\cap w(N),M} (\pi \otimes \sigma)) &= \\ &= \sum_i \sum_j \pi_i^{(1)} \times \pi_j^{(4)} \times \tilde{\pi}_i^{(3)} \otimes \pi_i^{(2)} \rtimes \sigma_j \\ &= \sum_i \sum_j \pi_i^{(1)} \times \tilde{\pi}_i^{(3)} \times \pi_j^{(4)} \otimes \pi_i^{(2)} \rtimes \sigma_j \\ &= \sum_i \sum_j \tilde{\pi}_i^{(3)} \times \pi_i^{(1)} \times \pi_j^{(4)} \otimes \pi_i^{(2)} \rtimes \sigma_j . \end{aligned}$$

PROOF. By Lemma 6.3, we have

$$N_{(i_1)} \cap w(N_{(i_2)}) = N_{(k,i_1-d,i_1,i_1+i_2-d-k)}.$$

It follows that

$$s.s.(r_{M\cap w(N),M} (\pi \otimes \sigma)) = \left(\sum_i \pi_i^{(1)} \otimes \pi_i^{(2)} \otimes \pi_i^{(3)}\right) \otimes \left(\sum_j \pi_j^{(4)} \otimes \sigma_j\right).$$

The above calculation gives $w^{-1}(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4 \otimes \sigma) = \pi_1 \otimes \pi_4 \otimes \tilde{\pi}_3 \otimes \pi_2 \otimes \sigma$. Since

$$w^{-1}(N_{(i_1)}) \cap N_{(i_2)} = N_{(k,i_2-d,i_2,i_1+i_2-d-k)},$$

we have $s.s.(i_{N,w^{-1}(M)\cap N} \circ w^{-1} \circ r_{M\cap w(N),M} (\pi \otimes \sigma)) =$

$$= \sum_i \sum_j \pi_i^{(1)} \times \pi_j^{(4)} \times \tilde{\pi}_i^{(3)} \otimes \pi_i^{(2)} \rtimes \sigma_j .$$

Now, we use the commutativity of R to obtain the other equalities. □

Define a $\mathbb{Z}$-bilinear mapping $\tilde{\rtimes} : (R \otimes R \otimes R) \times (R \otimes R(O)) \to R \otimes R(O)$ by defining

$$(\pi_1 \otimes \pi_2 \otimes \pi_3) \tilde{\rtimes} (\pi_4 \otimes \sigma) = \tilde{\pi}_1 \times \pi_2 \times \pi_4 \otimes \pi_3 \rtimes \sigma$$

for irreducible smooth representations π_i of $GL(n_i, F)$, $i = 1, 2, 3, 4$, and an irreducible smooth representation σ of $O(2m, F)$. Denote by s the homomorphism $s : R \otimes R \to R \otimes R$ which satisfies $s(r_1 \otimes r_2) = r_2 \otimes r_1$, $r_1, r_2 \in R$.

The proof of the following theorem uses calculations from [T1].

THEOREM 7.2. *Let π be an admissible finite length representation of $GL(i_1, F)$ and σ an admissible finite length representation of $O(2(n - i_1), F)$. Set*

$$\mathrm{m}^* = (1 \otimes m^*) \circ s \circ m^*.$$

Then,

$$\mu^*(\pi \rtimes \sigma) = \mathrm{m}^*(\pi) \tilde{\rtimes} \mu^*(\sigma).$$

PROOF. From [T1], we have

$$\mathfrak{m}^*(\pi)=\sum_{l=0}^{i_1}\left(\sum_{\substack{q,r\\0\le q\le i_1\\0\le r\le q\\i_1-q+r=l}}\left(\sum_{j=1}^{j_q}\sum_{u=1}^{u_r(j,q)}\beta_j^{(i_1-q)}\otimes\left(\gamma_j^{(q)}\right)_u^{(r)}\otimes\left(\delta_j^{(q)}\right)_u^{(q-r)}\right)\right),$$

$$\mu^*(\sigma)=\sum_{p=0}^{n-i_1}\left(\sum_{\nu=1}^{\nu_p}\tau_\nu^{(p)}\otimes\sigma_\nu^{(n-i_1-p)}\right),$$

$$\begin{aligned}\mathfrak{m}^*(\pi)\tilde{\rtimes}\mu^*(\sigma) &= \sum_{i_2=0}^{n}\left(\sum_{\substack{l,p\\0\le l\le i_1\\0\le p\le n-i_1\\l+p=i_2}}\ \sum_{\substack{q,r\\0\le q\le i_1\\0\le r\le q\\i_1-q+r=l}}\ \sum_{j=1}^{j_q}\sum_{u=1}^{u_r(j,q)}\sum_{\nu=1}^{\nu_p}\left(\beta_j^{(i_1-q)}\right)^\sim\right.\\ &\qquad\left.\times\left(\gamma_j^{(q)}\right)_u^{(r)}\times\tau_\nu^{(p)}\otimes\left(\delta_j^{(q)}\right)_u^{(q-r)}\rtimes\sigma_\nu^{(n-i_1-p)}\right).\end{aligned}$$

It is shown in [T1] that

(1)

$$\begin{aligned}&\mathfrak{m}^*(\pi)\tilde{\rtimes}\mu^*(\sigma)=\\ &\sum_{i_2=0}^{n}\left(\sum_{d=0}^{\min\{i_1,i_2\}}\ \sum_{k=\max\{0,(i_1+i_2-n)-d\}}^{\min\{i_1,i_2\}-d}\ \sum_{j=1}^{j_{i_1-d}}\ \sum_{u=1}^{u_k(j,i_1-d)}\ \sum_{\nu=1}^{\nu_{i_2-d-k}}\left(\beta_j^{(d)}\right)^\sim\right.\\ &\qquad\left.\times\left(\gamma_j^{(i_1-d)}\right)_u^{(k)}\times\tau_\nu^{(i_2-d-k)}\otimes\left(\delta_j^{(i_1-d)}\right)_u^{(i_1-d-k)}\rtimes\sigma_\nu^{(n-i_1-i_2+d+k)}\right)\end{aligned}$$

On the other hand, we have

$$\mu^*(\pi\rtimes\sigma)=\sum_{i_2=0}^{n}s.s.(s_{(i_2),(0)}(\pi\rtimes\sigma)).$$

Let $i_2\in\{1,\ldots,n\}$. Then

$s.s.(s_{(i_2),(0)}(\pi \rtimes \sigma)) = (\text{by Lemma 6.2}) =$

$$\sum_{d=0}^{\min\{i_1,i_2\}} \sum_{\substack{k=\max\{0,(i_1+i_2-n)-d\} \\ w=q_n(d,k)_{i_1,i_2}}}^{\min\{i_1,i_2\}-d} s.s.(i_{N_{(i_2)},w^{-1}(N_{(i_1)})\cap N_{(i_2)}} \circ w^{-1} \circ .$$

$$\circ\, r_{N_{(i_1)}\cap w(N_{(i_2)}),N_{(i_1)}}(\pi \otimes \sigma))$$

As in [T1], for d and k fixed, we have

$$\begin{aligned} s.s.(r_{(k,i_1-d-k,d),(i_1)}(\pi) &= \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_k(j,i_1-d)} \left(\gamma_j^{(i_1-d)}\right)_u^{(k)} \\ &\quad \otimes \left(\delta_j^{(i_1-d)}\right)_u^{(i_1-d-k)} \otimes \beta_j^{(d)}, \\ s.s.(s_{(i_2-d-k),(0)}(\sigma)) &= \sum_{\nu=1}^{\nu_{i_2-d-k}} \tau_\nu^{(i_2-d-k)} \otimes \sigma_\nu^{(n-i_1-i_2+d+k)}. \end{aligned}$$

Now, it follows from Lemma 7.1 that

$$\begin{aligned} &s.s.(s_{(i_2),(0)}(\pi \rtimes \sigma)) \\ &= \sum_{d=0}^{\min\{i_1,i_2\}} \sum_{k=\max\{0,(i_1+i_2-n)-d\}}^{\min\{i_1,i_2\}-d} \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_k(j,i_1-d)} \sum_{\nu=1}^{\nu_{i_2-d-k}} \left(\beta_j^{(d)}\right)^{\sim} \\ &\quad \times \left(\gamma_j^{(i_1-d)}\right)_u^{(k)} \times \tau_\nu^{(i_2-d-k)} \otimes \left(\delta_j^{(i_1-d)}\right)_u^{(i_1-d-k)} \rtimes \sigma_\nu^{(n-i_1-i_2+d+k)}. \end{aligned}$$

If $i_2 = 0$, then

$$s_{(i_2),(0)}(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma.$$

It follows

$$\begin{aligned} \mu^*(\pi \rtimes \sigma) &= \sum_{i_2=0}^{n} s.s.(s_{(i_2),(0)}(\pi \rtimes \sigma)) \\ &= \sum_{i_2=0}^{n} \Bigg(\sum_{d=0}^{\min\{i_1,i_2\}} \sum_{k=\max\{0,(i_1+i_2-n)-d\}}^{\min\{i_1,i_2\}-d} \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_k(j,i_1-d)} \sum_{\nu=1}^{\nu_{i_2-d-k}} \left(\beta_j^{(d)}\right)^{\sim} \\ &\quad \times \left(\gamma_j^{(i_1-d)}\right)_u^{(k)} \times \tau_\nu^{(i_2-d-k)} \otimes \left(\delta_j^{(i_1-d)}\right)_u^{(i_1-d-k)} \rtimes \sigma_\nu^{(n-i_1-i_2+d+k)} \Bigg). \end{aligned}$$

Now, the above equality and (1) give the theorem. □

For $r_1 \otimes r_2 \in R \otimes R$ and $r \otimes s \in R \otimes R(O)$, set

$$(r_1 \otimes r_2) \rtimes (r \otimes s) = (r_1 \times r) \otimes (r_2 \rtimes s).$$

Extend $\rtimes$ $\mathbb{Z}$-bilinearly to $\rtimes : (R \otimes R) \times (R \otimes R(O)) \to R \otimes R(O)$. Set

$$M^* = (m \otimes 1) \circ (\sim \otimes\, m^*) \circ s \circ m^*.$$

Theorem 7.2 now becomes

THEOREM 7.3. *For admissible finite length representations π of $GL(i_1, F)$ and σ of $O(2(n - i_1), F)$, we have*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

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