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ON WEIGHTED TURÁN TYPE INEQUALITY

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Let H_n be the class of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1,1]$, and R_n the class of real trigonometric polynomials of degree n with only real roots.

Define

$$\|f\| = \max_{-1\leq x\leq 1} |f(x)|.$$

In 1939, Turán [2] proved that for $f \in H_n$,

$$\|f'\| \geq C\sqrt{n}\|f\|,$$

this inequality was generalized by Varma [3] to L^2 norm, and by Zhou to general L^p norm, $0 < p \leq \infty$ (see [4]–[6]).

We notice that for orthogonal polynomials, the following result holds: If $\{\phi_n(x)\}$ is an orthogonal polynomial system on $[-1,1]$ with respect to a weight function[1] $W(x)$, then all zeros of any $\phi_n(x)$ are single and lie in the interval $(-1,1)$ (see [1: Chapter 4, §2]). Naturally, we should consider the potential applications of the Turán type inequality to orthogonal polynomial systems and thus raise the following conjecture.

Conjecture. *Let $f \in H_n$, then for $0 < p \leq \infty$ and some important weight functions $W(x)$ the inequality*

$$\left(\int_{-1}^{1} |f'(x)W(x)|^p dx\right)^{1/p} \geq C_W\sqrt{n}\left(\int_{-1}^{1} |f(x)W(x)|^p dx\right)^{1/p}$$

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[1] Without special needs, a weight function $W(x)$ is generally required to be nonnegative and measurable on $[-1,1]$.

holds, where the constant $C_W > 0$ depends upon $W(x)$ and p (when $0 < p < 1$) only.

Usually, to establish weighted inequalities for polynomials requires more complicated techniques, especially in general cases. The object of this paper is to start the work in uniform norm under general requirements for weight functions and establish the following

Theorem 1. *Let $W(x)$ be a nonnegative continuous piecewise monotone function*[2] *on the interval $[-1,1]$. If $f \in H_n$, then there exists a positive constant C_W only depending upon $W(x)$ such that*

$$\|f'W\| \geq C_W\sqrt{n}\|fW\|, \qquad (1)$$

where $\|\cdot\|$ indicates the uniform norm on $[-1,1]$.

Denote by $-1 \leq x_1 < x_2 < \cdots < x_k \leq 1$ all the distinct zeros of $f \in H_n$, and by l_i the multiplicity of x_i, $1 \leq i \leq k$, $x_0 = -1$, $x_{k+1} = 1$. According to the continuity of $W(x)$, we can find a $\delta > 0$ such for any local maximum point y_0 of $W(x)$ that[3] $W(x) \geq \frac{1}{2}W(y_0)$ in case $x \in (y_0 - \delta, y_0 + \delta) \cap [-1,1]$. Now in the sequel, we always suppose $n \geq 4/\delta^2$, that means, the absolute constants we get in the following proof for $n \geq 4/\delta^2$ must depend on $W(x)$ for all $n \geq 1$. Write the maximum point of $|f(x)|W(x)$ on $[-1,1]$ as α (note α may not be unique), that is,

$$|f(\alpha)|W(\alpha) = \|fW\|,$$

and without loss of generality, we may assume $f(\alpha) > 0$, and that $W(x)$ is non-decreasing in a small neighborhood of α, so $W(x)$ has a local maximum point $y_0 \geq \alpha$, and increases on $[\alpha, y_0]$. Suppose $x_r < \alpha < x_{r+1}$, $0 \leq r \leq k$, and β is the local maximum point of $|f(x)|$ in $[x_r, x_{r+1}]$. By the definition of α, we must have $\beta \leq \alpha$, otherwise α cannot be the maximum point of $|f(x)|W(x)$. If $\alpha = \beta = y_0$, this case will be treated as a lemma (we arrange it as Lemma 4), otherwise we have $r+1 \leq k$, or x_{r+1} is a zero of $f(x)$.

Set

$$m(x) = \sum_{i=1}^{k} \frac{l_i}{x - x_i},$$

[2]As usual, a function f is called piecewise monotone on $[-1,1]$ if there is a partition of $[-1,1]$: $-1 = a_0 < a_1 < \cdots < a_{s+1} = 1$ such that f is monotone on each $[a_j, a_{j+1}]$, $j = 0, 1, \cdots, s$.

[3]We note that such a δ only depends on the weight function $W(x)$.

obviously,

$$|m'(x)| = \sum_{i=1}^{k} \frac{l_i}{(x-x_i)^2} \geq \frac{n}{4}. \tag{2}$$

We divide the proof into several lemmas, and Theorem 1 follows by combining these lemmas.

Lemma 1. *If $y_0 - \alpha > n^{-1/2}$, then*

$$\|f'W\| \geq C_W \sqrt{n} \|fW\|.$$

Proof. We consider the following cases.

Case 1. $x_{r+1} \in (\alpha, \alpha + n^{-1/2}]$. In this case, by noting $x_{r+1} < y_0$ and $W(x) \geq W(\alpha)$, $x \in [\alpha, y_0]$, we calculate to deduce that

$$n^{-1/2}\|f'W\| \geq \int_{\alpha}^{x_{r+1}} |f'(x)|W(x)dx \geq W(\alpha) \int_{\alpha}^{x_{r+1}} |f'(x)|dx$$
$$= W(\alpha) \left| \int_{\alpha}^{x_{r+1}} f'(x)dx \right| = W(\alpha)(f(\alpha) - f(x_{r+1})) = W(\alpha)f(\alpha),$$

or inequality (1) holds.

Case 2. $x_{r+1} \notin (\alpha, \alpha + n^{-1/2}]$. By writing $\alpha^* = \alpha + n^{-1/2}$, we divide this case into subcases as follows (note $f(x) > 0$ for $x \in (\alpha, \alpha^*)$).

Subcase 2.1. $f(\alpha^*) < \frac{1}{2} f(\alpha)$. Similar argument to Case 1 leads to

$$n^{-1/2}\|f'W\| \geq \int_{\alpha}^{\alpha^*} |f'(x)|W(x)dx \geq W(\alpha)|f(\alpha) - f(\alpha^*)| \geq \frac{1}{2} W(\alpha)f(\alpha),$$

inequality (1) also holds.

Subcase 2.2. $f(\alpha^*) \geq \frac{1}{2} f(\alpha)$. Since $m(\beta) = 0$ and $\beta \leq \alpha$, we consider

$$W(\alpha^*)f'(\alpha^*) = W(\alpha^*)f(\alpha^*)m(\alpha^*) = W(\alpha^*)f(\alpha^*)(m(\alpha^*) - m(\beta))$$
$$= W(\alpha^*)f(\alpha^*)m'(\xi)(\alpha^* - \beta), \qquad \xi \in (\beta, \alpha^*).$$

In view of (2), the condition of this subcase and the definition of α^* ($\alpha^* - \beta \geq n^{-1/2}$), we get

$$\|f'W\| \geq \frac{1}{2} W(\alpha)f(\alpha)\frac{n}{4} n^{-1/2} = \frac{\sqrt{n}}{8} \|fW\|.$$

Combining Case 1 with Subcases 2.1 and 2.2, we have completed the proof of Lemma 1. □

Lemma 2. *If $y_0 - \alpha \leq n^{-1/2}$ and $\alpha - \beta > n^{-1/2}$, then*

$$\|f'W\| \geq C_W \sqrt{n} \|fW\|.$$

Proof. The proof of this lemma is very similar to that of Lemma 1. We give a sketch here.

Case 1. *x_r is a zero of $f(x)$ and $x_r \in [\alpha - 2^{-1}n^{-1/2}, \alpha)$.* In this case, we note $W(x) \geq \frac{1}{2}W(y_0)$ in case $x \in (y_0 - \delta, y_0 + \delta) \cap [-1,1]$, $n \geq 4/\delta^2$ and $0 < y_0 - x_r \leq \frac{3}{2\sqrt{n}}$, thus for any $x \in [x_r, \alpha]$,

$$W(x) \geq \frac{1}{2}W(y_0) \geq \frac{1}{2}W(\alpha), \qquad (3)$$

by similar proof we obtain

$$\|f'W\| \geq n^{1/2}\int_{x_r}^{\alpha}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{2}W(\alpha)f(\alpha).$$

Case 2. *x_r is not a zero of $f(x)$ or $x_r \notin [\alpha - 2^{-1}n^{-1/2}, \alpha)$.* Write $\alpha^* = \alpha - 2^{-1}n^{-1/2}$, we see $0 < y_0 - \alpha^* \leq \frac{3}{2}n^{-1/2}$, thus (3) holds for any $x \in [\alpha^*, \alpha]$.

Subcase 2.1. $f(\alpha^*) < \frac{1}{2}f(\alpha)$. Then

$$\|f'W\| \geq n^{1/2}\int_{\alpha^*}^{\alpha}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{8}W(\alpha)f(\alpha).$$

Subcase 2.2. $f(\alpha^*) \geq \frac{1}{2}f(\alpha)$. Now that $\alpha^* - \beta \geq 2^{-1}n^{-1/2}$, similar discussions will lead to

$$W(\alpha^*)|f'(\alpha^*)| = |W(\alpha^*)f(\alpha^*)(m(\alpha^*) - m(\beta))| \geq \frac{\sqrt{n}}{32}W(\alpha)f(\alpha).$$

All the cases show that Lemma 2 is true. □

Lemma 3. *If $y_0 - \alpha \leq n^{-1/2}$ and $\alpha - \beta \leq n^{-1/2}$, then*

$$\|f'W\| \geq C_W\sqrt{n}\|fW\|.$$

Proof. Setting $\alpha^* = \alpha + n^{-1/2}$, we see, $y_0 \in [\alpha, \alpha^*]$, and for all $x \in [\alpha, \alpha^*]$, (3) is valid. As before, we consider the following cases.

Case 1. $x_{r+1} \notin (\alpha, \alpha + n^{-1/2}]$. If $f(\alpha^*) < \frac{1}{2}f(\alpha)$, similarly to Subcase 2.1 of Lemma 1,

$$\|f'W\| \geq n^{1/2}\int_{\alpha}^{\alpha^*}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{4}W(\alpha)f(\alpha);$$

If $f(\alpha^*) \geq \frac{1}{2}f(\alpha)$, then similarly to Subcase 2.2 of Lemma 1 ($\alpha^* - \beta \geq n^{-1/2}$),

$$|W(\alpha^*)f'(\alpha^*)| = |W(\alpha^*)f(\alpha^*)(m(\alpha^*) - m(\beta))| \geq \frac{\sqrt{n}}{16}W(\alpha)f(\alpha).$$

Case 2. $x_{r+1} \in (\alpha, \alpha + n^{-1/2}]$. In view of $0 < x_{r+1} - y_0 \leq n^{-1/2}$, we still have inequality (3) for all $x \in [\alpha, x_{r+1}]$, therefore

$$\|f'W\| \geq n^{1/2}\int_{\alpha}^{x_{r+1}}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{2}W(\alpha)f(\alpha).$$

Altogether, Lemma 3 is finished. □

Finally, the argument of the case $\alpha = \beta = y_0$, which is quite similar to Lemma 3, will be proceeded as follows.

Lemma 4. *If* $\alpha = \beta = y_0$, *then*

$$\|f'W\| \geq C_W\sqrt{n}\|fW\|.$$

Proof. We see, either x_r or x_{r+1} must be a root of $f(x)$. Assume $f(x_r) = 0$, and set $\alpha^* = \alpha - n^{-1/2}$. As before, we consider the following cases.

Case 1. $x_r \notin (\alpha^*, \alpha]$. If $f(\alpha^*) < \frac{1}{2}f(\alpha)$, similarly to Subcase 2.1 of Lemma 1 (inequality (3) holds for any $x \in [\alpha^*, \alpha]$),

$$\|f'W\| \geq n^{1/2}\int_{\alpha^*}^{\alpha}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{2}W(\alpha)|f(\alpha) - f(\alpha^*)|$$
$$\geq \frac{\sqrt{n}}{4}W(\alpha)f(\alpha).$$

If $f(\alpha^*) \geq \frac{1}{2}f(\alpha)$, then similarly to Subcase 2.2 of Lemma 1 ($\beta - \alpha^* = \alpha - \alpha^* = n^{-1/2}$),

$$|W(\alpha^*)f'(\alpha^*)| = |W(\alpha^*)f(\alpha^*)(m(\alpha^*) - m(\beta))| \geq \frac{\sqrt{n}}{16}W(\alpha)f(\alpha).$$

Case 2. $x_r \in (\alpha^*, \alpha]$. In view of $0 < y_0 - x_r = \alpha - x_r \leq n^{-1/2}$, we still have inequality (3) for all $x \in [x_r, \alpha]$, therefore

$$\|f'W\| \geq n^{1/2}\int_{x_r}^{\alpha}|f'(x)|W(x)dx \geq \frac{\sqrt{n}}{2}W(\alpha)f(\alpha).$$

Lemma 4 is thus completed. □

To apply Theorem 1 to the important weight functions $(1+x)^\alpha(1-x)^\beta$, $\alpha, \beta \geq 0$, we have the following corollary.

Corollary 1. If $f \in H_n$, then there exists a positive constant $C_{\alpha,\beta}$ only depending upon α, β such that

$$\|f'(x)(1+x)^\alpha(1-x)^\beta\| \geq C_{\alpha,\beta}\sqrt{n}\|f(x)(1+x)^\alpha(1-x)^\beta\|, \quad \alpha, \beta \geq 0.$$

With $f(x) = (1-x^2)^{[n/2]}$, we see that the order $n^{1/2}$ in Corollary 1 can not be improved.

The corresponding result for trigonometric polynomials also holds.

Theorem 2. *Let* $W(x)$ *be a nonnegative continuous function of period* 2π *piecewise monotone on* $[0, 2\pi)$. *If* $f \in R_n$, *then there exists a positive constant* C_W *only depending upon* $W(x)$ *such that*

$$\|f'W\|_{C_{2\pi}} := \max_{-\infty<x<\infty}|f'(x)W(x)| \geq C_W\sqrt{n}\|fW\|_{C_{2\pi}}.$$

An application is

Corollary 2. If $f \in R_n$, then there exists a positive constant C_γ only depending upon γ such that

$$\| f'(x) |\sin x|^\gamma \|_{C_{2\pi}} \geq C_\gamma \sqrt{n}\, \| f(x) |\sin x|^\gamma \|_{C_{2\pi}}, \quad \gamma \geq 0.$$

REFERENCES

1. **E. W. Cheney,** *Introduction to Approximation Theory,* McGraw-Hill Book Co., 1966.
2. **P. Turán,** *Uber die ableitung von polynomen,* Compositio Math. 7(1939), 89-95.
3. **A. K. Varma,** *An analogue of some inequalities of P. Turán concerning algebraic polynomials satisfying certain conditions,* Proc. Amer. Math. Soc. 55(1976), 305-309.
4. **S. P. Zhou,** *Some remarks on Turán's inequality,* J. Approx. Theory 68(1992), 45-48.
5. **S. P. Zhou,** *Some remarks on Turán's inequality. II,* J. Math. Anal. Appl. 180(1993), 138-143.
6. **S. P. Zhou,** *Some remarks on Turán's inequality. III: The completion,* Anal. Math. 21(1995), 313-318.

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