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# CONVEX MAPPINGS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Let $B$ be the unit ball of $\mathbf{C}^n$ with respect to an arbitrary norm. We will give a sufficient condition for a local diffeomorphism of $C^1$ class on $B$ to be univalent and to have a convex image. Finally, we present an aplication on the complex ellipsoid $B(p_1, \dots, p_n)$, where $p_1, \dots, p_n \geq 1$.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mathbf{C}^n$ denote the space of $n$ complex variables $z = (z_1, \dots, z_n)'$ with an arbitrary norm $\|\cdot\|$.

Let $B$ be the unit ball of $\mathbf{C}^n$ with respect to this norm and also, let $B_r = rB$, for $0 < r \leq 1$. The symbol $'$ means the transpose of vectors and matrices.

By $L(\mathbf{C}^n, \mathbf{C}^m)$ we denote the space of continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^m$ with the standard operator norm. The letter $I$ means the identity in $L(\mathbf{C}^n, \mathbf{C}^n)$. The class of holomorphic mappings from a domain $G \subset \mathbf{C}^n$ into $\mathbf{C}^n$ is denoted by $H(G)$. If $f \in H(G)$, we define

$$Df(z) = \left[\frac{\partial f_j}{\partial z_k}(z)\right]_{1 \leq j,k \leq n}.$$

For a $C^1$ class mapping $f$ from a domain $G \subset \mathbf{C}^n$ into $\mathbf{C}^n$, let

$$J_r f(z) = \det \frac{\partial(u_1, v_1, \dots, u_n, v_n)}{\partial(x_1, y_1, \dots, x_n, y_n)},$$

where $z_j = x_j + \sqrt{-1}y_j$ and $f_j = u_j + \sqrt{-1}v_j$.

Suffridge [Su1], [Su2], Kikuchi [Ki] and Gong, Wang and Yu [Go-Wa-Yu2] gave analytic characterizations for locally biholomorphic mappings to be biholomorphic and convex.

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Suffridge [Su1], [Su2] and Gong, Wang and Yu [Go-Wa-Yu2] obtained some necessary and sufficient conditions of convexity for holomorphic mappings defined on the unit ball of $\mathbf{C}^n$ with respect to the Euclidean norm and an arbitrary norm. Also, Suffridge [Su2] characterized convex mappings on the unit ball of a complex Banach space, by a necessary and sufficient condition. On the other hand, Kikuchi [Ki] showed that Suffridge's results can be generalized to locally biholomorphic mappings on bounded domains in $\mathbf{C}^n$, for which the Bergman kernel function becomes infinite everywhere on the boundary.

Recently Hamada and Kohr [Ha-Ko1] gave necessary and sufficient conditions of convexity for locally biholomorphic mappings on bounded balanced pseudoconvex domains with $C^1$ plurisubharmonic defining functions.

In this paper we will obtain a sufficient condition of diffeomorphism and convexity for mappings of $C^1$ class on $B$. For other sufficient conditions of univalence on some domains in $\mathbf{C}^n$, see [Ko-Li].

## 2. Non-holomorphic case

We consider in this section a sufficient condition for a $C^1$ mapping from $B$ into $\mathbf{C}^n$, with $J_r f(z) \neq 0$, for $z \in B$, to be univalent and to have a convex image.

If $f \in C^1(B)$, we say that $f$ is convex if $f$ is univalent on $B$ and $f(B)$ is a convex domain.

Also, if $f \in C^1(B)$ with $f(0) = 0$, we say that $f$ is starlike if $f$ is univalent on $B$ and $f(B)$ is a starlike domain with respect to zero.

For each $z \in \mathbf{C}^n \setminus \{0\}$, let

$$T(z) = \{z^* \in L(\mathbf{C}^n, \mathbf{C}) : \ \|z^*\| = 1, \ z^*(z) = \|z\|\}.$$

Clearly, $T(z)$ is nonempty, by the Hahn-Banach theorem.

On the other hand, if $f \in C^1(B)$, $J_r f(z) \neq 0$, $z \in B$, then there exists a neighborhood $W_z$ of $z$ such that $f$ is a diffeormorphism of $C^1$ class between $W_z$ and $f(W_z)$, thus there exist the following matrices

$$D_w f^{-1}(w) = \left[\frac{\partial (f|_{W_z})_j^{-1}}{\partial w_k}(w)\right]_{1\leq j,k\leq n}$$

and

$$D_{\overline{w}} f^{-1}(w) = \left[\frac{\partial (f|_{W_z})_j^{-1}}{\partial \overline{w}_k}(w)\right]_{1\leq j,k\leq n}$$

where $w = f(z)$.

Recently Hamada and Kohr [Ha-Ko3] gave sufficient conditions for a local diffeomorphism of $C^1$ class on the unit ball of $\mathbf{C}^n$ with respect to an arbitrary norm to be univalent and to have a $\Phi$-like image. On the other hand, in [Ko2] and [Ha-Ko2], the authors obtained sufficient conditions of starlikeness and spirallikeness for mappings of $C^1$ class on the unit ball of $\mathbf{C}^n$ with a norm of $C^1$ class on $\mathbf{C}^n \setminus \{0\}$ and also, on bounded balanced pseudoconvex domains with $C^1$ plurisubharmonic defining functions.

In the following we prove a sufficient condition of diffeomorphism of $C^1$ class and convexity on the unit ball $B$ with respect to an arbitrary norm $\|\cdot\|$.

THEOREM 2.1. *Let $f \in C^1(B)$ such that $J_r f(z) \neq 0$, for all $z \in B$. If*

$$\text{(1)} \quad \operatorname{Re} z^* \left[ D_w f^{-1}(f(z))(f(z) - f(u)) + D_{\overline{w}} f^{-1}(f(z))(\overline{f(z)} - \overline{f(u)}) \right] > 0,$$

*for all $z, u \in B$, $\|u\| < \|z\|$, and $z^* \in T(z)$, then $f$ is convex.*

PROOF. Since $J_r f(z) \neq 0$, for all $z \in B$, then $f$ is a local diffeomorphism of $C^1$ class on $B$. We divide the proof into three steps, as follows.

First, we show that if $f$ is univalent on $B_r$, then $f$ is also univalent on $\overline{B}_r$, for all $r \in (0,1)$. If this assertion does not hold, then there exist at least two distinct points $z_1, z_2 \in \overline{B}_r$ such that $f(z_1) = f(z_2) = w$. Because $f$ is univalent on $B_r$ and $f$ is a local diffeomorphism, then $w \neq f(0)$. On the other hand, since $f$ is a local diffeomorphism on $B$, there exists a curve $z_1(s) : [-s_0, s_0] \to B \setminus \{0\}$, such that $z_1(s)$ is $C^1$ on $[-s_0, s_0]$, and

$$f(z_1(s)) = (1-s) f(z_1) + s f(0), \; s \in [-s_0, s_0],$$

with $z_1(0) = z_1$, for some $s_0 > 0$. Note that,

$$z_1(s) = z_1 - s w(z_1) + \epsilon(s), \; s \in (-s_0, s_0),$$

where $\lim\limits_{s \to 0} \dfrac{\epsilon(s)}{s} = 0$, and

$$w(x) = D_w f^{-1}(f(x))(f(x) - f(0)) + D_{\overline{w}} f^{-1}(f(x))(\overline{f(x)} - \overline{f(0)}),$$

for $x \in B \setminus \{0\}$. Taking into account the relation (1), for $u = 0$, we deduce that

$$\|z_1(s)\| \geq Re z_1^*(z_1(s)) = \|z_1\| - s Re z_1^*(w(z_1)) + \epsilon(s) > \|z_1\|,$$

for $s$ negative, such that $|s|$ is sufficiently small. Next, as in the proof of Theorem 2 of Suffridge [Su2], we conclude that $\|z_1(s)\|$ is strictly decreasing on $(-s_0, s_0)$, hence

$$\|z_1(s)\| < \|z_1(0)\| = \|z_1\| \leq r,$$

for all $s \in (0, s_0]$, so $z_1(s) \in B_r$, for all $s \in (0, s_0]$. Thus, we obtain the curve $z_1(s)$, which falls in $B_r$, for $0 < s \leq s_0$, such that $f(z_1(s)) = (1-s) f(z_1) + s f(0)$ and $z_1(0) = z_1$. Therefore, $z_1(s) = f^{-1}((1-s) f(z_1) + s f(0))$ is a univalent component of the inverse images of the curve $(1-s) f(z_1) + s f(0)$, for $0 \leq s \leq s_0$.

Suppose that $z_2(s)$ is another univalent component of the inverse images of the curve $(1-s) f(z_1) + s f(0)$, such that $z_2(s) \in B_r$, for sufficiently small $s > 0$, but with $z_2(0) = z_2$. Because $f$ is injective on $B_r$, $z_1(s) = z_2(s)$, for sufficiently small $s > 0$. However, this contradicts with the assumption $z_1(0) \neq z_2(0)$. Hence, we conclude that $f$ is also injective on $\overline{B}_r$.

In the second step we show that $\mathcal{M} = (0,1]$, where

$$\mathcal{M} = \{r \in (0,1] : f \text{ is injective on } B_r\}.$$

Since $J_r f(0) \neq 0$, there exists a small positive $\delta_1$ such that $f$ is a diffeomorphism of $C^1$ class from $B_{\delta_1}$ onto $f(B_{\delta_1})$. Therefore, $\mathcal{M}$ is nonempty.

We next show that $\mathcal{M}$ is closed.

If $0 < r_1 \in \mathcal{M}$, then all $r \in (0, r_1)$ fall in $\mathcal{M}$. Therefore, it suffices to show that if $r_1 > r$ and all $r \in \mathcal{M}$, then $r_1 \in \mathcal{M}$. If this assertion is not true, then there are at least two points $x_1$, $x_2 \in B_{r_1}$, such that $x_1 \neq x_2$, but, $f(x_1) = f(x_2)$. Since $\|x_1\| < r_1$, $\|x_2\| < r_1$, we can find a positive $r_2$ such that $\max\{\|x_1\|, \|x_2\|\} \leq r_2 < r_1$ and because $f$ is univalent on $B_{r_2}$, we deduce that $f(x_1) \neq f(x_2)$. So, we get a contradiction, thus we must have $\mathcal{M}$ closed.

Next, we show that $\mathcal{M}$ is open. To this end, it suffices to show that if $f$ is injective on $B_r$, then there exists a $\delta > 0$ small enough such that $f$ is also injective on $B_{r+\delta}$. If this is not true, then there is a sequence $(\delta_p)$, $\delta_p > 0$, $\lim_{p\to\infty} \delta_p = 0$, such that we can find two sequences $(x_p)$, $(y_p)$, which satisfy the following conditions

$$x_p, y_p \in B_{r+\delta_p}, x_p \neq y_p, f(x_p) = f(y_p),$$

for all $p = 1, 2, \ldots$.

Since $(x_p)$, $(y_p)$ are bounded sequences, there exist two subsequences $(x_{p_k})$, $(y_{p_k})$ of $(x_p)$ and $(y_p)$, such that

$$\lim_{k\to\infty} x_{p_k} = x, \ \lim_{k\to\infty} y_{p_k} = y$$

and also,

$$f(x_{p_k}) = f(y_{p_k}), \ x_{p_k} \neq y_{p_k}, \ k = 1, 2, \ldots.$$

Then $x, y \in \overline{B}_r$.

If $x \neq y$, this is contrary to the result of first step. If $x = y$, then there are two points $x'_{p_k} \in (x_{p_k})$, $y'_{p_k} \in (y_{p_k})$ in any neighborhood of $x = y$ such that $x'_{p_k} \neq y'_{p_k}$ and $f(x'_{p_k}) = f(y'_{p_k})$ and this is again a contradiction with $f$ locally univalent on $B$.

Hence $\mathcal{M}$ is a closed, open and nonempty subset of $(0,1]$, thus, $\mathcal{M} = (0,1]$.

In the last step we will show that $f(\overline{B}_r)$ is a convex set, for all $r \in (0,1)$, using a similar idea as in [Go-Wa-Yu1].

Let $r \in (0,1)$ and $x, y \in \overline{B}_r$. Let $\sigma(f(x), f(y))$ be the closed segment between $f(x)$ and $f(y)$. We will show that $\sigma(f(x), f(y)) \subset f(\overline{B}_r)$.

We may assume that $\|y\| < \|x\|$.

If we denote by $r(x,y) = \sigma(f(x), f(y)) \cap f(\overline{B}_r)$, then $r(x,y)$ is a closed set.

First, we show that there exists a $\delta_1 > 0$ such that

$$(1-t)f(x) + tf(y) \in f(\overline{B}_r), \tag{2}$$

for all $t \in (0, \delta_1)$.

For this aim, let $v(x,y,t) = f^{-1}((1-t)f(x)+tf(y))$, for $t \in [-\varepsilon, \varepsilon]$, where $\varepsilon$ is sufficiently small such that

$$(1-t)f(x) + tf(y) \in f(B), \quad t \in [-\varepsilon, \varepsilon].$$

Also, let

$$v(t) = v(x,y,t), \quad t \in [-\varepsilon, \varepsilon].$$

As in the first step of our proof, we can show that $||v(t)|| > ||x||$ when $t$ is negative, $|t|$ sufficiently small. However, using the fact that $v(v(x,y,\varepsilon), y, s) = v(x, y, s + (1-s)\varepsilon)$, we conclude that $||v(t)||$ is strictly decreasing on a neighborhood of zero.

Hence, the relation (2) holds.

Next, we show that $\sigma(f(x), f(y))$ is contained in $f(\overline{B}_r)$. For this aim, it suffices to show that $\nu(x,y) = \emptyset$, where

$$\nu(x,y) = \sigma(f(x), f(y)) \setminus r(x,y).$$

If we suppose that $\nu(x,y)$ is nonempty, then there exists

$$t^* = \inf\{t \in (0,1] : \ (1-t)f(x) + tf(y) \in \nu(x,y)\}.$$

Since $\nu(x,y) = \{\sigma(f(x), f(y)) \setminus \{f(x), f(y)\}\} \setminus r(x,y)$ is an open set, as a subset of $\sigma(f(x), f(y))$, then

$$Q(t^*) = (1-t^*)f(x) + t^* f(y) \notin \nu(x,y),$$

hence $Q(t^*) \in f(\overline{B}_r)$.

Let $z^* \in \overline{B}_r$ such that $f(z^*) = Q(t^*)$. If $||z^*|| > ||y||$, then the result of the first step can be applied and there exists a $\delta_2 > 0$ such that

$$(1-t)Q(t^*) + tf(y) \in f(\overline{B}_r),$$

for all $t \in (0, \delta_2)$. However, when $||z^*|| \le ||y||$, then the result of the first step cannot be applied. In this case we can show the existence of $\delta_2$ directly.

Therefore,

$$(1 - t - t^* + tt^*)f(x) + (t + t^* - tt^*)f(y) \in f(\overline{B}_r),$$

for all $t \in (0, \delta_2)$.

However, this is contrary to the definition of the infimum, hence we conclude that $\nu(x,y) = \emptyset$.

So, $\sigma(f(x), f(y)) \subset f(\overline{B}_r)$, as desired.

Since $f(B) = \bigcup_{0<r<1} f(\overline{B}_r)$ and $f(\overline{B}_r)$ is a convex set, for all $r \in (0,1)$, then $f(B)$ is also a convex domain. This completes the proof.

REMARK 2.1. We note that if $f$ is holomorphic in Theorem 2.1, then we obtain a similar sufficient condition of convexity, as in the finite dimensional case of Theorem 5 of Suffridge [Su2].

Also, if in Theorem 2.1 we assume that $f(0) = 0$, then we can obtain a similar sufficient condition of starlikeness on the unit ball $B$ (cf. [Ko2], [Ha-Ko2], [Ha-Ko3]).

On the other hand, let $D$ be a bounded balanced convex domain in $\mathbf{C}^n$. Also, let $h$ be the Minkowski function of $D$. Then, it is well known that $h$ is a norm on $\mathbf{C}^n$ and $D$ is the unit ball with respect to this norm (see [Ja-Pf]). Then Theorem 2.1 holds for $D$.

For example, let $B(p_1, \dots, p_n)$ be the complex ellipsoid, where $p_1, \dots, p_n \geq 1$ and

$$B(p_1, \dots, p_n) = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^{n} |z_j|^{p_j} < 1 \right\},$$

for $z = (z_1, \dots, z_n)' \in \mathbf{C}^n$. Then $B(p_1, \dots, p_n)$ is a balanced convex domain in $\mathbf{C}^n$.

Also, if $f \in C^1(B(p_1, \dots, p_n))$, let

$$w(x,y) = D_w f^{-1}(f(x))(f(x) - f(y)) + D_{\overline{w}} f^{-1}(f(x))(\overline{f(x)} - \overline{f(y)}),$$

for $x, y \in B(p_1, \dots, p_n)$.

In this case, we obtain the following result.

COROLLARY 2.1. *Let $f \in C^1(B(p_1, \dots, p_n))$ such that $J_r f(z) \neq 0$, for all $z \in B(p_1, \dots, p_n)$, where $p_1, \dots, p_m > 1$, $p_{m+1} = \dots = p_n = 1$.*

*If*

$$Re \sum_{j=1}^{m} p_j w_j(x,y) \frac{|x_j|^{p_j}}{h^{p_j}(x) x_j} + Re \sum_{\substack{j=m+1 \\ x_j \neq 0}}^{n} w_j(x,y) \frac{|x_j|}{h(x) x_j} - \sum_{\substack{j=m+1 \\ x_j = 0}}^{n} \frac{|w_j(x,y)|}{h(x)} > 0,$$

*for all $x = (x_1, \dots, x_n)'$ and $y = (y_1, \dots, y_n)' \in B(p_1, \dots, p_n)$, with $h(y) < h(x)$, then $f$ is convex.*

PROOF. We can use a similar idea as in the section 3 of [Su1].

Using similar notations and reasons as in the proof of Theorem 2.1, we wish to conclude that for given $x, y \in B(p_1, \dots, p_n)$, with $h(y) < h(x)$, then

$$h(v(x,y,t)) < h(x), \text{ for } 0 < t < \epsilon,$$

for some $\epsilon > 0$. This will be true provided

$$\sum_{j=1}^{n} \frac{|x_j - t w_j(x,y)|^{p_j}}{h^{p_j}(x)} < \sum_{j=1}^{n} \frac{|x_j|^{p_j}}{h^{p_j}(x)},$$

for $t$ sufficiently small. That is

$$\sum_{\substack{j=1\\ x_j\neq 0}}^{n} \frac{|x_j|^{p_j}}{h^{p_j}(x)}\Big(1-2tRew_j(x,y)/x_j+t^2|w_j(x,y)/x_j|^2\Big)^{p_j/2}+$$

$$+\sum_{\substack{j=1\\ x_j=0}}^{n} \frac{t^{p_j}}{h^{p_j}(x)}|w_j(x,y)|^{p_j} < \sum_{j=1}^{n}\frac{|x_j|^{p_j}}{h^{p_j}(x)}.$$

This condition is satisfied when

$$t\Big(\sum_{\substack{j=1\\ x_j\neq 0}}^{n} -p_j\frac{|x_j|^{p_j}}{h^{p_j}(x)}Re\frac{w_j(x,y)}{x_j}+\sum_{\substack{j=m+1\\ x_j=0}}^{n} t^{p_j-1}\frac{|w_j(x,y)|^{p_j}}{h^{p_j}(x)}\Big)<0,$$

where $t$ is sufficiently small positive. Therefore, if

$$Re\sum_{j=1}^{m} p_jw_j(x,y)\frac{|x_j|^{p_j}}{x_jh^{p_j}(x)}+Re\sum_{\substack{j=m+1\\ x_j\neq 0}}^{n} w_j(x,y)\frac{|x_j|}{h(x)x_j}-\sum_{\substack{j=m+1\\ x_j=0}}^{n}\frac{|w_j(x,y)|}{h(x)}>0,$$

for $x=(x_1,\dots,x_n)'$, $y=(y_1,\dots,y_n)'\in B(p_1,\dots,p_n)$, with $h(y)<h(x)$, then, taking into account the proof of Theorem 2.1, we conclude that $f$ is convex. This completes the proof. □

Further on, let

$$\langle z,w\rangle=\sum_{j=1}^{n} z_j\overline{w}_j,$$

denote the inner scalar product on the Euclidean space $\mathbf{C}^n$, then from Theorem 2.1, we obtain the following consequence.

COROLLARY 2.2. *Let $f\in C^1(B)$ such that $J_rf(z)\neq 0$, for all $z\in B$. If*

$$\mathrm{Re}\,\Big\langle D_wf^{-1}(f(z))(f(z)-f(u))+D_{\overline{w}}f^{-1}(f(z))(\overline{f(z)}-\overline{f(u)}),z\Big\rangle>0,$$

*for all $z,u\in B$, $\|u\|<\|z\|$, then $f$ is convex.*

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