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FIRST ORDER DIFFERENTIAL EQUATIONS WITH A PARAMETER

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ABSTRACT. Employing the method of upper and lower solutions and monotone iterative technique, existence of extremal solutions to differential equations with a parameter is proved.

1. INTRODUCTION

We concentrate our attention on the following differential equation

$$x'(t) = f(t, x(t), \lambda), \quad t \in J = [0, b] \tag{1}$$

with the conditions:

$$x(0) = k_0, \qquad G(x(b), \lambda) = 0, \tag{2}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $G \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $k_0 \in \mathbb{R}$ are given. By a solution of problem (1)–(2) we mean a pair $(x, \lambda) \in C^1(J, \mathbb{R}) \times \mathbb{R}$ for which (1)–(2) is satisfied. Problem (1)–(2) is called a problem with a parameter. Problems with a parameter have been considered for many years. Some of them appeared as mathematical model of physical systems (see, for example [**7**]).

The important area of research in the qualitative theory of differential equations is study of existence of solutions. Existence theorems can be formulated under the assumption that f and G satisfy the Lipschitz condition with respect to the last two variables with suitable Lipschitz constants or Lipschitz functions (see, for example [**1**], [**2**], [4], [**6**]). The purpose of this

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paper is to formulate an existence theorem for problem (1)–(2) employing the method of upper and lower solutions. This method gives a solution in a closed set. Using this technique, we construct monotone sequences giving sufficient conditions under which they are convergent. It is important to add that the one-sided Lipschitz condition is assumed on f and G. This paper extends the result of paper [**3**], where it was assumed that f is nondecreasing with respect to the last variable.

2. Main result

A pair $(v,\alpha) \in C^1(J,\mathbb{R}) \times \mathbb{R}$ is said to be a lower solution of (1)–(2) if

$$\begin{cases} v'(t) & \leq & f(t,v(t),\alpha), \; t \in J, \\ v(0) & \leq & k_0, \\ 0 & \leq & G(v(b),\alpha), \end{cases}$$

and an upper solution of (1)–(2) if the above inequalities are reversed.

Theorem 1. *Assume that $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $G \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and*

1^o *$(y_0,\lambda_0),(z_0,\gamma_0) \in C^1(J,\mathbb{R}) \times \mathbb{R}$ are lower and upper solutions of problem (1)–(2) such that $y_0(t) \leq z_0(t)$ on J, and $\lambda_0 \leq \gamma_0$,*

2^o *f is nondecreasing with respect to the last variable,*

3^o *$f(t,\bar{u},\lambda) - f(t,u,\lambda) \geq -M(\bar{u}-u)$ for $y_0 \leq u \leq \bar{u} \leq z_0$ with $M \geq 0$,*

4^o *G is nondecreasing with respect to the first variable,*

5^o *$G(u,\bar{\lambda}) - G(u,\lambda) \geq -N(\bar{\lambda}-\lambda)$ for $\lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0$ with $N > 0$.*

Then there exist monotone sequences $\{y_n,\lambda_n\}$, $\{z_n,\gamma_n\}$ such that $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$ and $\lambda_n \to \lambda$, $\gamma_n \to \gamma$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, (y,λ) and (z,γ) are minimal and maximal solutions of problem (1)–(2), respectively.

Proof. For $k = 0,1,\cdots$, we construct monotone sequences by formulas:

$$\begin{cases} y'_{k+1}(t) & = & f(t,y_k(t),\lambda_k) - M[y_{k+1}(t) - y_k(t)], \quad y_{k+1}(0) = k_0, \\ 0 & = & G(y_k,\lambda_k) - N(\lambda_{k+1} - \lambda_k), \end{cases}$$

and

$$\begin{cases} z'_{k+1}(t) & = & f(t,z_k(t),\gamma_k) - M[z_{k+1}(t) - z_k(t)], \quad z_{k+1}(0) = k_0, \\ 0 & = & G(z_k,\gamma_k) - N(\gamma_{k+1} - \gamma_k). \end{cases}$$

First of all, we shall prove that

$$\begin{cases} \lambda_0 \leq \lambda_1 \leq \gamma_1 \leq \gamma_0, \\ y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J. \end{cases} \tag{3}$$

Put $p = \lambda_0 - \lambda_1$. Then, we have

$$0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) \geq -N(\lambda_1 - \lambda_0) = Np,$$

so $p \leq 0$ and hence $\lambda_0 \leq \lambda_1$. Now, let $p = \lambda_1 - \gamma_1$. In view of $1^o, 4^o$ and 5^o, we have

$$\begin{aligned} 0 &= G(y_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) \\ &\leq G(z_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) \\ &\leq N(\gamma_0 - \lambda_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) = -Np. \end{aligned}$$

Hence $\lambda_1 \leq \gamma_1$. Note that if $p = \gamma_1 - \gamma_0$, then

$$0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0) \leq -N(\gamma_1 - \gamma_0) = -Np,$$

and hence $\gamma_1 \leq \gamma_0$. As a result, we have the first part of (3).

Let $p(t) = y_0(t) - y_1(t),\ t \in J$. In view of 1^o, we see that

$$\begin{aligned} p'(t) &= y_0'(t) - y_1'(t) \leq f(t, y_0(t), \lambda_0) - f(t, y_0(t), \lambda_0) + M[y_1(t) - y_0(t)] \\ &= -Mp(t), \quad t \in J, \end{aligned}$$

and $p(0) = y_0(0) - y_1(0) \leq 0$. It shows that $p(t) \leq 0,\ t \in J$, so $y_0(t) \leq y_1(t),\ t \in J$. Put $p(t) = y_1(t) - z_1(t),\ t \in J$. Then, in view of $1^o, 2^o$ and 3^o, we have

$$\begin{aligned} p'(t) &= y_1'(t) - z_1'(t) \\ &= f(t, y_0(t), \lambda_0) - M[y_1(t) - y_0(t)] - f(t, z_0(t), \gamma_0) \\ &\quad + M[z_1(t) - z_0(t)] \\ &\leq f(t, y_0(t), \gamma_0) - f(t, z_0(t), \gamma_0) - M[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\leq M[z_0(t) - y_0(t)] - M[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &= -Mp(t), \quad t \in J, \end{aligned}$$

and $p(0) = 0$, so $p(t) \leq 0,\ t \in J$, and $y_1(t) \leq z_1(t),\ t \in J$. Put $p(t) = z_1(t) - z_0(t),\ t \in J$. Then, by 1^o, we obtain

$$\begin{aligned} p'(t) &= z_1'(t) - z_0'(t) \leq f(t, z_0(t), \gamma_0) - M[z_1(t) - z_0(t)] - f(t, z_0(t), \gamma_0) \\ &= -Mp(t), \quad t \in J \quad \text{with} \quad p(0) \leq 0, \end{aligned}$$

so $p(t) \leq 0,\ t \in J$, and hence $z_1(t) \leq z_0(t),\ t \in J$. This shows that (3) is satisfied.

In the next step, we are going to show that (y_1, λ_1) and (z_1, γ_1) are lower and upper solutions of problem (1)–(2). Note that

$$\begin{aligned} y_1'(t) &= f(t, y_0(t), \lambda_0) - M[y_1(t) - y_0(t)] \\ &= f(t, y_1(t), \lambda_1) + f(t, y_0(t), \lambda_0) - f(t, y_1(t), \lambda_1) \\ &\quad - M[y_1(t) - y_0(t)] \\ &\leq f(t, y_1(t), \lambda_1) + f(t, y_0(t), \lambda_1) - f(t, y_1(t), \lambda_1) \\ &\quad - M[y_1(t) - y_0(t)] \\ &\leq f(t, y_1(t), \lambda_1) + M[y_1(t) - y_0(t)] \\ &\quad - M[y_1(t) - y_0(t)] \\ &= f(t, y_1(t), \lambda_1), \quad t \in J, \quad y_1(0) = k_0, \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &= f(t,z_0(t),\gamma_0) - M[z_1(t)-z_0(t)] \\ &= f(t,z_1(t),\gamma_1) + f(t,z_0(t),\gamma_0) - f(t,z_1(t),\gamma_1) - M[z_1(t)-z_0(t)] \\ &\geq f(t,z_1(t),\gamma_1) + f(t,z_0(t),\gamma_1) - f(t,z_1(t),\gamma_1) - M[z_1(t)-z_0(t)] \\ &\geq f(t,z_1(t),\gamma_1) - M[z_0(t)-z_1(t)] - M[z_1(t)-z_0(t)] \\ &= f(t,z_1(t),\gamma_1), \quad t \in J, \quad z_1(0) = k_0. \end{aligned}$$

Moreover, in view of 4^o and 5^o, we have

$$\begin{aligned} 0 &= G(y_0,\lambda_0) - N(\lambda_1-\lambda_0) \leq G(y_1,\lambda_0) - N(\lambda_1-\lambda_0) \\ &= G(y_1,\lambda_0) - G(y_1,\lambda_1) + G(y_1,\lambda_1) - N(\lambda_1-\lambda_0) \\ &\leq N(\lambda_1-\lambda_0) + G(y_1,\lambda_1) - N(\lambda_1-\lambda_0) = G(y_1,\lambda_1), \end{aligned}$$

and

$$\begin{aligned} 0 &= G(z_0,\gamma_0) - N(\gamma_1-\gamma_0) \geq G(z_1,\gamma_0) - N(\gamma_1-\gamma_0) \\ &= G(z_1,\gamma_0) - G(z_1,\gamma_1) + G(z_1,\gamma_1) - N(\gamma_1-\gamma_0) \\ &\geq -N(\gamma_0-\gamma_1) + G(z_1,\gamma_1) - N(\gamma_1-\gamma_0) = G(z_1,\gamma_1). \end{aligned}$$

By the above considerations, (y_1,λ_1) and (z_1,γ_1) are lower and upper solutions of (1)–(2).

Let us assume that

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k \leq \gamma_k \leq \gamma_{k-1} \leq \cdots \leq \gamma_1 \leq \gamma_0,$$

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_{k-1}(t) \leq y_k(t)$$
$$\leq z_k(t) \leq z_{k-1}(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J$$

and

$$\begin{cases} y_k'(t) \leq f(t,y_k(t),\lambda_k), & y_k(0) = k_0, \\ 0 \leq G(y_k,\lambda_k), \end{cases}$$

$$\begin{cases} z_k'(t) \geq f(t,z_k(t),\gamma_k), & z_k(0) = k_0, \\ 0 \geq G(z_k,\gamma_k) \end{cases}$$

for some $k > 1$. We shall prove that

$$\text{(4)} \qquad \begin{cases} \lambda_k \leq \lambda_{k+1} \leq \gamma_{k+1} \leq \gamma_k, \\ y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J, \end{cases}$$

and

$$\begin{cases} y_{k+1}'(t) \leq f(t,y_{k+1}(t),\lambda_{k+1}), & y_{k+1}(0) = k_0, \\ 0 \leq G(y_{k+1},\lambda_{k+1}), \end{cases}$$

$$\begin{cases} z_{k+1}'(t) \geq f(t,z_{k+1}(t),\gamma_{k+1}), & z_{k+1}(0) = k_0, \\ 0 \geq G(z_{k+1},\gamma_{k+1}). \end{cases}$$

Put $p = \lambda_k - \lambda_{k+1}$, so

$$0 = G(y_k,\lambda_k) - N(\lambda_{k+1}-\lambda_k) \geq Np,$$

and hence $\lambda_k \le \lambda_{k+1}$. Let $p = \lambda_{k+1} - \gamma_{k+1}$. Then, in view of 4^o and 5^o, we see that

$$\begin{aligned} 0 &= G(y_k, \lambda_k) - G(z_k, \gamma_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1} - \gamma_k) \\ &\le G(z_k, \lambda_k) - G(z_k, \gamma_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1} - \gamma_k) \\ &\le N(\gamma_k - \lambda_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1} - \gamma_k) = -Np. \end{aligned}$$

Hence we have $\lambda_{k+1} \le \gamma_{k+1}$. Now, let $p = \gamma_{k+1} - \gamma_k$. Then

$$0 = G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k) \le -Np,$$

so $\gamma_{k+1} \le \gamma_k$, which shows that the first inequality of (4) is satisfied.

Similarly as before, we can show that $y_k(t) \le y_{k+1}(t)$, and $z_{k+1}(t) \le z_k(t)$, $t \in J$. Note that for $p(t) = y_{k+1}(t) - z_{k+1}(t)$, $t \in J$, we obtain

$$\begin{aligned} p'(t) &= f(t, y_k(t), \lambda_k) - M[y_{k+1}(t) - y_k(t)] - f(t, z_k(t), \lambda_k) \\ &\quad + M[z_{k+1}(t) - z_k(t)] \\ &\le f(t, y_k(t), \gamma_k) - f(t, z_k(t), \gamma_k) \\ &\quad - M[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &\le M[z_k(t) - y_k(t)] - M[y_{k+1}(t) - y_k(t) \\ &\quad - z_{k+1}(t) + z_k(t)] \\ &= -Mp(t), \quad t \in J, \text{ and } \quad p(0) = 0. \end{aligned}$$

It proves that $y_{k+1}(t) \le z_{k+1}(t)$, $t \in J$, so $y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t)$, $t \in J$, and hence, (4) holds.

Now we are going to show that (y_{k+1}, λ_{k+1}) and (z_{k+1}, γ_{k+1}) are lower and upper solutions of problem (1)–(2). Indeed, we see that

$$\begin{aligned} y'_{k+1}(t) &= f(t, y_k(t), \lambda_k) - M[y_{k+1}(t) - y_k(t)] \\ &= f(t, y_{k+1}(t), \lambda_{k+1}) + f(t, y_k(t), \lambda_k) \\ &\quad - f(t, y_{k+1}(t), \lambda_{k+1}) - M[y_{k+1}(t) - y_k(t)] \\ &\le f(t, y_{k+1}(t), \lambda_{k+1}) + f(t, y_k(t), \lambda_{k+1}) \\ &\quad - f(t, y_{k+1}(t), \lambda_{k+1}) - M[y_{k+1}(t) - y_k(t)] \\ &\le f(t, y_{k+1}(t), \lambda_{k+1}) + M[y_{k+1}(t) - y_k(t)] \\ &\quad - M[y_{k+1}(t) - y_k(t)] \\ &= f(t, y_{k+1}(t), \lambda_{k+1}) \text{ with } \quad y_{k+1}(0) = k_0, \end{aligned}$$

and

$$\begin{aligned} z'_{k+1}(t) &= f(t, z_k(t), \gamma_k) - M[z_{k+1}(t) - z_k(t)] \\ &= f(t, z_{k+1}(t), \gamma_{k+1}) + f(t, z_k(t), \gamma_k) \\ &\quad - f(t, z_{k+1}(t), \gamma_{k+1}) - M[z_{k+1}(t) - z_k(t)] \\ &\ge f(t, z_{k+1}(t), \gamma_{k+1}) + f(t, z_k(t), \gamma_{k+1}) \\ &\quad - f(t, z_{k+1}(t), \gamma_{k+1}) - M[z_{k+1}(t) - z_k(t)] \\ &\ge f(t, z_{k+1}(t), \gamma_{k+1}) - M[z_k(t) - z_{k+1}(t)] - M[z_{k+1}(t) - z_k(t)] \\ &= f(t, z_{k+1}(t), \gamma_{k+1}) \text{ with } \quad z_{k+1}(0) = k_0. \end{aligned}$$

Moreover, in view of 4^o and 5^o, we have

$$\begin{aligned} 0 &= G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \leq G(y_{k+1}, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \\ &= G(y_{k+1}, \lambda_k) - G(y_{k+1}, \lambda_{k+1}) + G(y_{k+1}, \lambda_{k+1}) - N(\lambda_{k+1} - \lambda_k) \\ &\leq N(\lambda_{k+1} - \lambda_k) + G(y_{k+1}, \lambda_{k+1}) - N(\lambda_{k+1} - \lambda_k) = G(y_{k+1}, \lambda_{k+1}), \end{aligned}$$

and

$$\begin{aligned} 0 &= G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k) \geq G(z_{k+1}, \gamma_k) - N(\gamma_{k+1} - \gamma_k) \\ &= G(z_{k+1}, \gamma_k) - G(z_{k+1}, \gamma_{k+1}) + G(z_{k+1}, \gamma_{k+1}) - N(\gamma_{k+1} - \gamma_k) \\ &\geq -N(\gamma_k - \gamma_{k+1}) + G(z_{k+1}, \gamma_{k+1}) - N(\gamma_{k+1} - \gamma_k) = G(z_{k+1}, \gamma_{k+1}). \end{aligned}$$

It proves that (y_{k+1}, λ_{k+1}), (z_{k+1}, γ_{k+1}) are lower and upper solutions of problem (1)–(2).

Hence, by induction, we have

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \gamma_n \leq \cdots \leq \gamma_1 \leq \gamma_0,$$

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), t \in J$$

for all n. Emploing standard techniques (see [5]), it can be shown that the sequences $\{y_n, \lambda_n\}$, $\{z_n, \gamma_n\}$ converge uniformly and monotonically to (y, λ), (z, γ), respectively. Indeed, (y, λ) and (z, γ) are solutions of problem (1)–(2) in view of the continuity of f and G, and the definitions of the above sequences.

Now, we need to prove that if (u, β) is any solution of problem (1)–(2) such that

$$y_0(t) \leq u(t) \leq z_0(t), \quad t \in J, \text{ and } \lambda_0 \leq \beta \leq \gamma_0,$$

then the following inequalities

$$y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t), \quad t \in J, \qquad \text{and} \qquad \lambda_0 \leq \lambda \leq \beta \leq \gamma \leq \gamma_0$$

are satisfied.

First, let $p(t) = y_1(t) - u(t)$, $t \in J$. Then

$$\begin{aligned} p'(t) &= y_1'(t) - u'(t) = f(t, y_0(t), \lambda_0) - M[y_1(t) - y_0(t)] - f(t, u(t), \beta) \\ &\leq f(t, y_0(t), \beta) - f(t, u(t), \beta) - M[y_1(t) - y_0(t)] \\ &\leq M[u(t) - y_0(t)] - M[y_1(t) - y_0(t)] = -Mp(t) \text{ with } p(0) = 0, \end{aligned}$$

so $y_1(t) \leq u(t)$, $t \in J$. Now, let $p(t) = u(t) - z_1(t)$, $t \in J$. Then

$$\begin{aligned} p'(t) &= u'(t) - z_1'(t) = f(t, u(t), \beta) - f(t, z_0(t), \gamma_0) + M[z_1(t) - z_0(t)] \\ &\leq f(t, u(t), \gamma_0) - f(t, z_0(t), \gamma_0) + M[z_1(t) - z_0(t)] \\ &\leq M[z_0(t) - u(t)] + M[z_1(t) - z_0(t)] = -Mp(t) \text{ with } p(0) = 0, \end{aligned}$$

and hence $u(t) \leq z_1(t)$, $t \in J$.

Put $p = \lambda_1 - \beta$. Then

$$\begin{aligned} 0 &= G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) \leq G(u, \lambda_0) - N(\lambda_1 - \lambda_0) \\ &= G(u, \lambda_0) - G(u, \beta) - N(\lambda_1 - \lambda_0) \\ &\leq N(\beta - \lambda_0) - N(\lambda_1 - \lambda_0) = -Np, \end{aligned}$$

so $p \le 0$, and hence $\lambda_1 \le \beta$. Now, we put $p = \beta - \gamma_1$. Then

$$\begin{aligned} 0 &= G(u,\beta) \le G(z_0,\beta) = G(z_0,\beta) - G(z_0,\gamma_0) + N(\gamma_1 - \gamma_0) \\ &\le N(\gamma_0 - \beta) + N(\gamma_1 - \gamma_0) = -Np, \end{aligned}$$

and $p \le 0$ which means that $\beta \le \gamma_1$. From the above we have

$$y_0(t) \le y_1(t) \le u(t) \le z_1(t) \le z_0(t), \ \ t \in J, \ \text{ and } \ \lambda_0 \le \lambda_1 \le \beta \le \gamma_1 \le \gamma_0.$$

Let us assume that

$$y_k(t) \le u(t) \le z_k(t), \ \ t \in J, \qquad \text{and} \qquad \lambda_k \le \beta \le \gamma_k$$

for some $k > 1$. Put $p = \lambda_{k+1} - \beta$. Then, in view of 4^o and 5^o, we have

$$\begin{aligned} 0 &= G(y_k,\lambda_k) - N(\lambda_{k+1} - \lambda_k) \le G(u,\lambda_k) - N(\lambda_{k+1} - \lambda_k) \\ &= G(u,\lambda_k) - G(u,\beta) - N(\lambda_{k+1} - \lambda_k) \\ &\le N(\beta - \lambda_k) - N(\lambda_{k+1} - \lambda_k) = -Np, \end{aligned}$$

so $p \le 0$ and hence $\lambda_{k+1} \le \beta$. Let $p = \beta - \gamma_{k+1}$. Then we obtain

$$\begin{aligned} 0 &= G(u,\beta) \le G(z_k,\beta) = G(z_k,\beta) - G(z_k,\gamma_k) + N(\gamma_{k+1} - \gamma_k) \\ &\le N(\gamma_k - \beta) + N(\gamma_{k+1} - \gamma_k) = -Np, \end{aligned}$$

and hence $p \le 0$, so $\beta \le \gamma_{k+1}$. This shows that

$$\lambda_{k+1} \le \beta \le \gamma_{k+1}.$$

As before, we set $p(t) = y_{k+1}(t) - u(t), \ t \in J$. Then, in view of 2^o and 3^o, we obtain

$$\begin{aligned} p'(t) &= y'_{k+1} - u'(t) = f(t, y_k(t), \lambda_k) \\ &\quad -M[y_{k+1}(t) - y_k(t)] - f(t, u(t), \beta) \\ &\le f(t, y_k(t), \beta) - f(t, u(t), \beta) - M[y_{k+1}(t) - y_k(t)] \\ &\le M[u(t) - y_k(t)] \\ &\quad -M[y_{k+1}(t) - y_k(t)] = -Mp(t), \quad t \in J \ \text{ with } \ p(0) = 0, \end{aligned}$$

hence $p(t) \le 0, \ t \in J$, and $y_{k+1}(t) \le u(t), \ t \in J$. Put $p(t) = u(t) - z_{k+1}(t), \ t \in J$. Indeed, in this case, we have

$$\begin{aligned} p'(t) &= u'(t) - z'_{k+1}(t) = f(t, u(t), \beta) - f(t, z_k(t), \gamma_k) \\ &\quad +M[z_{k+1}(t) - z_k(t)] \\ &\le f(t, u(t), \gamma_k) - f(t, z_k(t), \gamma_k) \\ &\quad +M[z_{k+1} - z_k(t)] \\ &\le M[z_k(t) - u(t)] + \\ &\quad M[z_{k+1}(t) - z_k(t)] \le -Mp(t) \ \text{ with } \ p(0) = 0. \end{aligned}$$

Hence $p(t) \le 0, \ t \in J$, so $u(t) \le z_{k+1}(t), \ t \in J$. This shows that

$$y_{k+1}(t) \le u(t) \le z_{k+1}(t), \ \ t \in J.$$

By induction, this proves that the inequalities

$$y_n(t) \le u(t) \le z_n(t), \ \ t \in J, \qquad \text{and} \qquad \lambda_n \le \beta \le \gamma_n$$

are satisfied for all n. Taking the limit as $n \to \infty$, we conclude that

$$y(t) \le u(t) \le z(t), \quad t \in J, \qquad \text{and} \qquad \lambda \le \beta \le \gamma.$$

It means that (y, λ), (z, γ) are minimal and maximal solutions of (1)–(2). This completes the proof of the theorem. □

Now we are going to prove some relations between the members of sequences from Theorem 1 and sequences defined below by formulas:

$$\begin{cases} \bar{y}'_{k+1}(t) &= f(t, \bar{y}_k(t), \bar{\lambda}_k) - P[\bar{y}_{k+1}(t) - \bar{y}_k(t)], \\ & \quad \bar{y}_{k+1}(0) = k_0, \ \bar{y}_0(t) = y_0(t), \quad t \in J, \\ 0 &= G(\bar{y}_k, \bar{\lambda}_k) - Q(\bar{\lambda}_{k+1} - \bar{\lambda}_k), \\ & \quad \bar{\lambda}_0 = \lambda_0, \end{cases}$$

$$\begin{cases} \bar{z}'_{k+1}(t) &= f(t, \bar{z}_k(t), \bar{\gamma}_k) - P[\bar{z}_{k+1}(t) - \bar{z}_k(t)], \\ & \quad \bar{z}_{k+1}(0) = k_0, \ \bar{z}_0(t) = z_0(t), \ t \in J, \\ 0 &= G(\bar{z}_k, \bar{\gamma}_k) - Q(\bar{\gamma}_{k+1} - \bar{\gamma}_k), \\ & \quad \bar{\gamma}_0 = \gamma_0 \end{cases}$$

for $k = 0, 1, \cdots$.

LEMMA 1. *Let the assumptions of Theorem 1 be satisfied. If $M \le P$, $N \le Q$, then*

$$\begin{cases} \bar{\lambda}_n \le \lambda_n \le \gamma_n \le \bar{\gamma}_n, \\ \bar{y}_n(t) \le y_n(t) \le z_n(t) \le \bar{z}_n(t), \ t \in J \end{cases} \tag{5}$$

for $n = 0, 1, \cdots$.

PROOF. Note that the relations: $\lambda_n \le \gamma_n$, $y_n(t) \le z_n(t)$, $t \in J$, $n = 0, 1, \cdots$ follow from Theorem 1.

Let $p = \bar{y}_1 - y_1$. Then

$$\begin{aligned} p'(t) &= f(t, y_0(t), \lambda_0) - P[\bar{y}_1(t) - y_0(t)] \\ &\quad -f(t, y_0(t), \lambda_0) + M[y_1(t) - y_0(t)] \\ &= -P[\bar{y}_1(t) - y_1(t)] \\ &\quad +(M - P)[y_1(t) - y_0(t)] \le -Pp(t), \ p(0) = 0, \end{aligned}$$

which proves that $\bar{y}_1(t) \le y_1(t)$, $t \in J$. If we now put $q = \bar{\lambda}_1 - \lambda_1$, then

$$\begin{aligned} 0 &= G(y_0, \lambda_0) - Q(\bar{\lambda}_1 - \lambda_0) - G(y_0, \lambda_0) + N(\lambda_1 - \lambda_0) \\ &= -Q(\bar{\lambda}_1 - \lambda_1) + (N - Q)(\lambda_1 - \lambda_0) \le -Qq, \end{aligned}$$

so $\bar{\lambda}_1 \le \lambda_1$. Similarly, we can show that $z_1(t) \le \bar{z}_1(t)$, $t \in J$, $\gamma_1 \le \bar{\gamma}_1$. It means that (5) holds for n=1.

Now we assume that (5) is satisfied for $n = k$. Put $p = \bar{y}_{k+1} - y_{k+1}$, so $p(0) = 0$. Then, by the assumptions 2^o and 3^o of Theorem 1, we get

$$\begin{aligned} p'(t) &= f(t, \bar{y}_k(t), \bar{\lambda}_k) - P[\bar{y}_{k+1}(t) - \bar{y}_k(t)] \\ &\quad -f(t, y_k(t), \lambda_k) + M[y_{k+1}(t) - y_k(t)] \\ &\leq f(t, \bar{y}_k(t), \lambda_k) - f(t, y_k(t), \lambda_k) \\ &\quad -P[\bar{y}_{k+1}(t) - \bar{y}_k(t)] + M[y_{k+1}(t) - y_k(t)] \\ &\leq M[y_k(t) - \bar{y}_k(t)] - \\ &\quad P[\bar{y}_{k+1}(t) - y_{k+1}(t) + y_{k+1}(t) - \bar{y}_k(t)] \\ &\quad +M[y_{k+1}(t) - y_k(t)] \\ &= -Pp(t) + (M - P)[y_{k+1}(t) - y_k(t) \\ &\quad +y_k(t) - \bar{y}_k(t)] \leq -Pp(t), \end{aligned}$$

so $p(t) \leq 0$ on J, and hence $\bar{y}_{k+1}(t) \leq y_{k+1}(t)$ on J.

If we put $q = \bar{\lambda}_{k+1} - \lambda_{k+1}$, then, in view of assumptions 4^o and 5^o of Theorem 1, we get

$$\begin{aligned} 0 &= G(\bar{y}_k, \bar{\lambda}_k) - Q(\bar{\lambda}_{k+1} - \bar{\lambda}_k) - G(y_k, \lambda_k) + N(\lambda_{k+1} - \lambda_k) \\ &\leq G(y_k, \bar{\lambda}_k) - G(y_k, \lambda_k) - Q(\bar{\lambda}_{k+1} - \bar{\lambda}_k) + N(\lambda_{k+1} - \lambda_k) \\ &\leq N(\lambda_k - \bar{\lambda}_k) - Q(\bar{\lambda}_{k+1} - \bar{\lambda}_k) + N(\lambda_{k+1} - \lambda_k) \\ &= -Qq + (N - Q)(\lambda_{k+1} - \lambda_k + \lambda_k - \bar{\lambda}_k) \leq -Qq, \end{aligned}$$

so $q \leq 0$, and hence $\bar{\lambda}_{k+1} \leq \lambda_{k+1}$.

Similarly, for $p = z_{k+1} - \bar{z}_{k+1}$, we obtain

$$\begin{aligned} p'(t) &= f(t, z_k(t), \gamma_k) \\ &\quad -M[z_{k+1}(t) - z_k(t)] - f(t, \bar{z}_k, \bar{\gamma}_k) \\ &\quad +P[\bar{z}_{k+1}(t) - \bar{z}_k(t)] \\ &\leq f(t, z_k(t), \bar{\gamma}_k) - f(t, \bar{z}_k(t), \bar{\gamma}_k) \\ &\quad -M[z_{k+1}(t) - z_k(t)] + P[\bar{z}_{k+1}(t) - \bar{z}_k(t)] \\ &\leq M[\bar{z}_k(t) - z_k(t)] - M[z_{k+1}(t) - z_k(t)] \\ &\quad +P[\bar{z}_{k+1}(t) - \bar{z}_k(t)] \leq -Pp(t), \quad p(0) = 0, \end{aligned}$$

and as the result we have $z_{k+1}(t) \leq \bar{z}_{k+1}(t)$ on J. Moreover, if $q = \gamma_{k+1} - \bar{\gamma}_{k+1}$, then

$$\begin{aligned} 0 &= G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k) - G(\bar{z}_k, \bar{\gamma}_k) + Q(\bar{\gamma}_{k+1} - \bar{\gamma}_k) \\ &\leq G(\bar{z}_k, \gamma_k) - G(\bar{z}_k, \bar{\gamma}_k) - N(\gamma_{k+1} - \gamma_k) + Q(\bar{\gamma}_{k+1} - \bar{\gamma}_k) \\ &\leq N(\bar{\gamma}_k - \gamma_k) - N(\gamma_{k+1} - \gamma_k) + Q(\bar{\gamma}_{k+1} - \bar{\gamma}_k) \leq -Qq, \end{aligned}$$

so $\gamma_{k+1} \leq \bar{\gamma}_{k+1}$.

By the above and mathematical induction, we see that (5) is satisfied. This ends the proof. □

3. Remarks

REMARK 1. We observe that the special case when f is monotone nondecreasing with respect to the second variable is covered by our theorem. To see this, it is enought to put $M = 0$ in condition 3^o.

REMARK 2. If we assume that G is nondecreasing with respect to the second variable, then there exists $N > 0$ such that for $\bar{\lambda} \geq \lambda$ we have

$$G(u, \bar{\lambda}) - G(u, \lambda) \geq 0 = 0(\bar{\lambda} - \lambda) \geq -N(\bar{\lambda} - \lambda).$$

This shows that condition 5^o holds.

REMARK 3. Note that, by 1^o and 4^o, we obtain

$$G(y_0, \gamma_0) \leq G(z_0, \gamma_0) \leq 0 \leq G(y_0, \lambda_0),$$

so

$$0 \leq G(y_0, \lambda_0) - G(y_0, \gamma_0).$$

Moreover, if G is also nondecreasing with respect to the second variable, then

$$0 \leq G(y_0, \lambda_0) \leq G(y_0, \gamma_0) \leq G(z_0, \gamma_0) \leq 0,$$

$$0 \leq G(y_0, \lambda_0) \leq G(z_0, \lambda_0) \leq G(z_0, \gamma_0) \leq 0,$$

so

$$G(y_0, \lambda_0) = G(y_0, \gamma_0) = G(z_0, \lambda_0) = G(z_0, \gamma_0) = 0.$$

In the same way we can show that

$$G(y_n, \lambda_n) = G(y_n, \gamma_n) = G(z_n, \lambda_n) = G(z_n, \gamma_n) = 0, \quad n = 0, 1, \cdots.$$

It proves that in assumptions of Theorem 1, function G can not be increasing with respect to the second variable on the whole interval $[\lambda_0, \gamma_0]$, but it can be increasing only on some subintervals of $[\lambda_0, \bar{\beta}]$ and $[\bar{\beta}, \gamma_0]$, where $(\bar{y}, \bar{\beta})$ is the root of the equation $G(y, \lambda) = 0$.

REMARK 4. Let

$$G(u, \lambda) = G(\lambda) = \begin{cases} -\sin\lambda, & \lambda \in [-\frac{\pi}{2}, \pi - 1], \\ -\dfrac{\lambda + 1 - \pi}{1 + \pi} - \sin(\pi - 1), & \lambda \in (\pi - 1, 2\pi]. \end{cases}$$

Note that G is continuous on $[-\frac{\pi}{2}, 2\pi]$, and it is increasing on $(\frac{\pi}{2}, \pi - 1)$. Condition 5^o is satisfied with $N = 1$. Note that $\lambda = 0$ is the unique solution of the equation $G(\lambda) = 0$. To find this solution we can apply the method of monotone iterations. Put $\lambda_0 = -\frac{\pi}{2}$, $\gamma_0 = 2\pi$. Then $\lambda_0 < \gamma_0$ and $G(\lambda_0) = 1 > 0$, $G(\gamma_0) \approx -1.8415 < 0$, so λ_0 and γ_0 are lower and upper solutions of the equation $G(\lambda) = 0$.

Below, in the table, there are some values of $\{\lambda_n, \gamma_n\}$:

n	λ_n	γ_n	$G(\lambda_n)$	$G(\gamma_n)$
0	−1.5708	6.2832	1.0000	−1.8415
1	−0.5708	4.4417	0.5403	−1.3968
2	−0.0305	3.0449	0.0305	−1.0596
3	0.0000	1.9853	0.0000	−0.9153
4		1.0700		−0.8772
5		0.1928		−0.1916
6		0.0012		−0.0012
7		0.0000		0.0000

Indeed, $\lambda_n \to 0$, $\gamma_n \to 0$, so $\lambda = 0$ is the unique solution of $G(\lambda) = 0$.

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