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FUNCTIONAL BOUNDARY VALUE PROBLEMS WITHOUT GROWTH RESTRICTIONS

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ABSTRACT. Let $J = [0,T]$ and $F : C^0(J) \times C^0(J) \times \mathbb{R} \to L_1(J)$ be an operator. Existence theorems for the functional differential equation $(g(x'(t)))' = (F(x, x', x'(t)))(t)$ with functional boundary conditions generalizing the non-homogeneous Dirichlet boundary conditions and non-homogeneous mixed boundary conditions are given. Existence results are proved by the Leray–Schauder degree theory under some sign conditions imposed upon F.

1. INTRODUCTION

Let $J = [0, T]$ be a compact interval. Consider the functional differential equation

$$(g(x'(t)))' = (F(x, x', x'(t)))(t). \tag{1}$$

Here $g : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$, $g(0) = 0$ and $F : C^0(J) \times C^0(J) \times \mathbb{R} \to L_1(J)$, $(x, y, a) \longmapsto (F(x, y, a))(t)$ is an operator having the following properties:

(a) $(F(x, y, z(t)))(t) \in L_1(J)$ for $x, y, z \in C^0(J)$,

(b) $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$ in $C^0(J) \times C^0(J) \times C^0(J) \Rightarrow$
$\lim_{n\to\infty}(F(x_n, y_n, z_n(t)))(t) = (F(x, y, z(t)))(t)$ in $L_1(J)$,

(c) for each $d \in (0, \infty)$ there exists $k_d \in L_1(J)$, such that $x, y \in C^0(J)$, $a \in \mathbb{R}$, $\|x\| + \|y\| + |a| \leq d \Rightarrow |(F(x, y, a))(t)| \leq k_d(t)$ for a.e. $t \in J$,

where $\|x\| = \max\{|x(t)|; t \in J\}$ for $x \in C^0(J)$ is the norm in $C^0(J)$.

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A prototype of the operator F in (1) is the operator

$$(F(x,y,a))(t) = f(t,x(t),y(t),a)$$

where $f : J \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions on $J \times \mathbb{R}^3$ ($f \in Car(J \times \mathbb{R}^3)$ for short) or more generally

$$(F(x,y,a))(t) = (P_1(x,y))(t)h(a) + (P_2(x,y))(t)$$

and

$$(F(x,y,a))(t) = \int_t^{T-t} f_1(s,ax(s),y(s),a)\,ds + f_2(t,x(t),y(t),a)$$

where P_1, $P_2 : C^0(J) \times C^0(J) \to L_1(J)$, $h : \mathbb{R} \to \mathbb{R}$ are continuous and, for each $d \in (0,\infty)$, there exists $l_d \in L_1(J)$ such that x, $y \in C^0(J)$, $\|x\|+\|y\| \le d \Rightarrow |(P_1(x,y))(t)| \le l_d(t)$, $|(P_2(x,y))(t)| \le l_d(t)$ for a.e. $t \in J$ and $f_1, f_2 \in Car(J \times \mathbb{R}^3)$.

Together with (1) consider the functional boundary conditions

$$x(\alpha_1(x,x')) = p_1(x,x'), \quad x(\alpha_2(x,x')) = p_2(x,x'), \tag{2}$$

or

$$x(\beta_1(x,x')) = r_1(x,x'), \quad x'(\beta_2(x,x')) = r_2(x,x'). \tag{3}$$

Here α_1, α_2, β_1, $\beta_2 : C^0(J)\times C^0(J) \to J$ and p_1, p_2, r_1, $r_2 : C^0(J)\times C^0(J) \to \mathbb{R}$ are continuous functionals. We see that (2) (with $\alpha_1(x,x') = 0$, $\alpha_2(x,x') = T$, $p_1(x,x') = A$, $p_2(x,x') = B$ for $x \in C^1(J)$) gives the nonhomogeneous Dirichlet boundary conditions and (3) (with $r_1(x,x') = A$, $r_2(x,x') = B$ and $\beta_1(x,x') = 0$, $\beta_2(x,x') = T$ resp. $\beta_1(x,x') = T$, $\beta_2(x,x') = 0$ for $x \in C^1(J)$) gives the nonhomogeneous mixed boundary conditions.

We say that $x \in C^1(J)$ is a *solution of the boundary value problem* (BVP for short) (1), (j) ($j = 2, 3$) if $g(x'(t))$ is absolutely continuous on J, x satisfies boundary conditions (j) and (1) is satisfied for a.e. $t \in J$.

We observe that Brykalov [B] considered among others the differential equation

$$x'' + a_1(t)x' + a_0(t)x = f(t,x,x')$$

together with boundary conditions (2) (actually with more general boundary conditions, in which α_i, p_i can depend also on x''). For this BVP he proved an existence result under the assumptions that $a_0, a_1 \in L_1(J)$, $a_0(t) \le 0$, $f \in Car(J\times\mathbb{R}^2)$ satisfies the growth condition $|f(t,x,y)| \le \gamma(t)+A_0|x|^{1-\varepsilon_0} + A_1|y|^{1-\varepsilon_1}$ for a.e. $t \in J$ and each $x, y \in \mathbb{R}$ where $\gamma \in L_1(J)$, $\gamma(t) \ge 0$, $A_i \in (0,\infty)$, $\varepsilon_i \in (0,1)$ ($i = 0,1$) and $|p_1(x,x')-p_2(x,x')| \le \lambda|\alpha_1(x,x')-\alpha_2(x,x')|$, $|p_j(x,x')| \le N$ ($j = 1,2$) for all x having the absolutely continuous derivative on J with positive constants λ and N.

In this paper we prove existence results for BVPs (1), (2) and (1), (3) providing that F satisfies only sign conditions. Our results are proved by the topological degree method (see e.g. [D] and [M]). We generalize the results of

[K] for the Dirichlet conditions where the differential equation $x'' = h(t, x, x')$, $h \in C^0(J \times \mathbb{R}^2)$ was studied. We note that our results are close those of **[RT]** for the Dirichlet conditions where another type of the functional differential equation was considered. This functional differential equation without growth restrictions and with nonlinear functional boundary conditions was considered in **[S]**. Some existence results for the equation $x'' = h(t, x, x')$ with continuous h without growth restrictions was given by Rodriguez and Tineo **[RT]** for the Dirichlet problem and by Ruyun Ma **[R]** for an m-point boundary value problem.

The following assumptions will be needed throughout the paper:

(H_1) $\alpha_1(x, x') < \alpha_2(x, x'), \quad x \in C^1(J)$;

(H_2) There exists a positive constant μ such that

$$|p_1(x, x') - p_2(x, x')| \le \mu(\alpha_2(x, x') - \alpha_1(x, x')), \quad x \in C^1(J);$$

(H_3) There exist positive constants A_1, A_2 such that

$$|p_i(x, x')| \le A_i, \quad x \in C^1(J), \quad i = 1, 2;$$

(H_4) There exist $L_1, L_2, L_3, L_4 \in \mathbb{R}$ such that $L_1, L_4 \in (-\infty, -\mu]$, $L_2, L_3 \in [\mu, \infty)$, $L_1 \ne L_4$, $L_2 \ne L_3$ and

$$(F(x, y, L_1))(t) \le 0 \le (F(x, y, L_2))(t),$$

$$(F(x, y, L_3))(t) \le 0 \le (F(x, y, L_4))(t)$$

for a.e. $t \in J$ and each $x, y \in C^0(J)$, $\|x\| \le U$, $D \le y(t) \le H$ for $t \in J$, where

$$U = \min\{A_1, A_2\} + T\max\{-D, H\}, \quad D = \min\{L_1, L_4\},$$

$$H = \max\{L_2, L_3\}.$$

(H_5) There exist positive constants M, N such that

$$|r_1(x, x')| \le M, \quad |r_2(x, x')| \le N, \quad x \in C^1(J);$$

(H_6) There exist $K_1, K_2, K_3, K_4 \in \mathbb{R}$ such that $K_1, K_4 \in (-\infty, -N]$, $K_2, K_3 \in [N, \infty)$, $K_1 \ne K_4$, $K_2 \ne K_3$ and

$$(F(x, y, K_1))(t) \le 0 \le (F(x, y, K_2))(t),$$

$$(F(x, y, K_3))(t) \le 0 \le (F(x, y, K_4))(t)$$

for a.e. $t \in J$ and each $x, y \in C^0(J)$, $\|x\| \le U_*$, $D_* \le y(t) \le H_*$ for $t \in J$, where

$$U_* = M + T\max\{-D_*, H_*\}, \quad D_* = \min\{K_1, K_4\},$$

$$H_* = \max\{K_2, K_3\}.$$

2. BVP (1), (2)

Assume that assumptions (H_1)–(H_4) are satisfied. Let $|L_4 - L_1| > \frac{2}{n_0}$, $|L_3 - L_2| > \frac{2}{n_0}$ for an $n_0 \in \mathbb{N}$. Set

$$E_1 = L_1 + \tfrac{\operatorname{sign}(L_4 - L_1) - 1}{2}(L_1 - L_4), \quad E_2 = L_2 + \tfrac{\operatorname{sign}(L_3 - L_2) - 1}{2}(L_2 - L_3),$$
$$E_3 = L_3 - \tfrac{\operatorname{sign}(L_3 - L_2) - 1}{2}(L_2 - L_3), \quad E_4 = L_4 - \tfrac{\operatorname{sign}(L_4 - L_1) - 1}{2}(L_1 - L_4).$$

Then $E_1 < E_4 \le -\mu$, $\mu \le E_2 < E_3$ and $D = E_1$, $H = E_3$.

For each $n \ge n_0$, $x, y \in C^0(J)$ and $a \in \mathbb{R}$, define $\bar{x}, \tilde{y} \in C^0(J)$ and $[a]_n \in \mathbb{R}$ by

$$\bar{x}(t) = \begin{cases} U & \text{for } x(t) > U \\ x(t) & \text{for } |x(t)| \le U \\ -U & \text{for } x(t) < -U, \end{cases}$$

$$\tilde{y}(t) = \begin{cases} E_3 & \text{for } y(t) > E_3 \\ y(t) & \text{for } E_1 \le y(t) \le E_3 \\ E_1 & \text{for } y(t) < E_1, \end{cases}$$

$$(4) \qquad [a]_n = \begin{cases} E_3 & \text{for } a \ge E_3 \\ a & \text{for } E_2 + \frac{2}{n} < a < E_3 \\ -E_2 + 2a - \frac{2}{n} & \text{for } E_2 + \frac{1}{n} < a \le E_2 + \frac{2}{n} \\ E_2 & \text{for } E_2 < a \le E_2 + \frac{1}{n} \\ a & \text{for } E_4 \le a \le E_2 \\ E_4 & \text{for } E_4 - \frac{1}{n} \le a < E_4 \\ -E_4 + 2a + \frac{2}{n} & \text{for } E_4 - \frac{2}{n} \le a < E_4 - \frac{1}{n} \\ a & \text{for } E_1 \le a < E_4 - \frac{2}{n} \\ E_1 & \text{for } a < E_1. \end{cases}$$

Clearly $\lim_{n\to\infty}[a]_n = a$ for $a \in [E_1, E_3]$ and for any $z \in C^0(J)$, $E_1 \le z(t) \le E_3$, we have $\lim_{n\to\infty}[z(t)]_n = z(t)$ uniformly on J.

Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property:

$$(5) \qquad \begin{gathered} |p(v)| \le 1 \quad \text{for } v \in \mathbb{R}, \\ p(v) = 1 \quad \text{for } v \in [L_4 - \tfrac{1}{n_0}, L_4] \cup [L_2, L_2 + \tfrac{1}{n_0}], \\ p(v) = -1 \quad \text{for } v \in [L_1 - \tfrac{1}{n_0}, L_1] \cup [L_3, L_3 + \tfrac{1}{n_0}]. \end{gathered}$$

Set

$$(F_n(x, y, a))(t) = (F(\bar{x}, \tilde{y}, [a]_n))(t) + \frac{p(a)}{n}$$

for $(x, y, a) \in C^0(J) \times C^0(J) \times \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq n_0$.

Consider the two–parameter family of the functional differential equations

$$(g(x'(t)))' = \lambda(F_n(x, x', x'(t)))(t), \qquad \lambda \in [0,1],\ n \geq n_0. \tag{6_n)$_\lambda$}$$

LEMMA 2.1. (A priori estimates). *Let assumptions* $(H_1) - (H_4)$ *be satisfied with* $L_1 < L_4$ *and* $L_2 < L_3$ *and let BVP* $(6_n)_\lambda$, (2) *has a solution* u *for some* $\lambda \in [0,1]$ *and* $n \geq n_0$. *Then the estimates*

$$\|u\| \leq U + \frac{T}{n}, \quad L_1 - \frac{1}{n} < u'(t) < L_3 + \frac{1}{n}$$

for $t \in J$ *are fulfilled.*

PROOF. Set $t_1 = \alpha_1(u, u')$, $t_2 = \alpha_2(u, u')$. Then (H_1), (H_2) and (H_3) imply $t_1 < t_2$, $|u(t_2) - u(t_1)| = |p_2(u, u') - p_1(u, u')| \leq \mu(t_2 - t_1)$, and so $\frac{|u(t_2)-u(t_1)|}{t_2-t_1} \leq \mu$. Hence

$$|u'(\xi)| \leq \mu \tag{7}$$

where ξ lies between t_1 and t_2. If $\lambda = 0$ then $g(u'(t)) \equiv const.$, and so (cf. (7))

$$|u'(t)| = |u'(\xi)| \leq \mu, \qquad t \in J.$$

Let $\lambda \in (0,1]$. Let $u'(T_1) = \max\{u'(t); t \in J\} \geq L_3 + \frac{1}{n}$ with a $T_1 \in J$. Assume $T_1 \in (\xi, T]$. Then there exist $t_* \in (\xi, T_1)$ and $\varepsilon_* > 0$ such that $u'(t_*) = L_3$, $u'(t_* + \varepsilon_*) = L_3 + \frac{1}{n}$ and $L_3 \leq u'(t) \leq L_3 + \frac{1}{n}$ for $t \in [t_*, t_* + \varepsilon_*]$. Integrating the equality

$$(g(u'(t)))' = \lambda(F_n(u, u', u'(t)))(t) \tag{8}$$

for a.e. $t \in J$ from t_* to $t_* + \varepsilon_*$ we obtain

$$\begin{aligned} g(u'(t_* + \varepsilon_*)) - g(u'(t_*)) &= \lambda \int_{t_*}^{t_*+\varepsilon_*} (F_n(u, u', u'(t)))(t)\, dt \\ &= \lambda \int_{t_*}^{t_*+\varepsilon_*} \left((F(\bar{u}, \widetilde{u'}, L_3))(t) + \frac{p(u'(t))}{n} \right) dt \\ &\leq \frac{\lambda}{n} \int_{t_*}^{t_*+\varepsilon_*} p(u'(t))\, dt = -\frac{\lambda \varepsilon_*}{n} < 0, \end{aligned}$$

which contradicts $g(u'(t_* + \varepsilon_*)) - g(u'(t_*)) = g(L_3 + \frac{1}{n}) - g(L_3) > 0$. Assume $T_1 \in [0, \xi)$. Then there exist $t_0 \in (T_1, \xi]$ and $\varepsilon_0 > 0$ such that $u'(t_0 - \varepsilon_0) = L_2 + \frac{1}{n}$, $u'(t_0) = L_2$ and $L_2 \leq u'(t) \leq L_2 + \frac{1}{n}$ for $t \in [t_0 - \varepsilon_0, t_0]$. Integrating (8) from $t_0 - \varepsilon_0$ to t_0 we have

$$g(u'(t_0)) - g(u'(t_0 - \varepsilon_0)) = \lambda \int_{t_0-\varepsilon_0}^{t_0} (F_n(u, u', u'(t)))(t)\, dt$$

$$= \lambda \int_{t_0-\varepsilon_0}^{t_0} \Big((F(\bar{u}, \widetilde{u'}, L_2))(t) + \frac{p(u'(t))}{n} \Big)\, dt$$
$$\geq \frac{\lambda}{n} \int_{t_0-\varepsilon_0}^{t_0} p(u'(t))\, dt = \frac{\lambda \varepsilon_0}{n} > 0,$$

which contradicts $g(u'(t_0)) - g(u'(t_0 - \varepsilon_0)) = g(L_2) - g(L_2 + \frac{1}{n}) < 0$. Hence $u'(t) < L_3 + \frac{1}{n}$ for $t \in J$.

Let $u'(T_2) = \min\{u'(t);\, t \in J\} \leq L_1 - \frac{1}{n}$ for some $T_2 \in J$. Assume $T_2 \in (\xi, T]$. Then there exist $t_+ \in [\xi, T_2)$ and $\varepsilon_+ > 0$ such that $u'(t_+) = L_4$, $u'(t_+ + \varepsilon_+) = L_4 - \frac{1}{n}$ and $L_4 - \frac{1}{n} \leq u'(t) \leq L_4$ for $t \in [t_+, t_+ + \varepsilon_+]$. Integrating (8) from t_+ to $t_+ + \varepsilon_+$ we obtain

$$g(u'(t_+ + \varepsilon_+)) - g(u'(t_+)) = \lambda \int_{t_+}^{t_+ + \varepsilon_+} (F_n(u, u', u'(t)))(t)\, dt$$
$$= \lambda \int_{t_+}^{t_+ + \varepsilon_+} \Big((F(\bar{u}, \widetilde{u'}, L_4))(t) + \frac{p(u'(t))}{n} \Big)\, dt$$
$$\geq \frac{\lambda}{n} \int_{t_+}^{t_+ + \varepsilon_+} p(u'(t))\, dt = \frac{\lambda \varepsilon_+}{n} > 0$$

which contradicts $g(u'(t_+ + \varepsilon_+)) - g(u'(t_+)) = g(L_4 - \frac{1}{n}) - g(L_4) < 0$. If $T_2 \in [0, \xi)$ then there exist $t_- \in (T_2, \xi)$ and $\varepsilon_- > 0$ such that $u'(t_- - \varepsilon_-)) = L_1 - \frac{1}{n}$, $u'(t_-) = L_1$, $L_1 - \frac{1}{n} \leq u'(t) \leq L_1$ for $t \in [t_- - \varepsilon_-, t_-]$. Integrating (8) from $t_- - \varepsilon_-$ to t_- we have

$$g(u'(t_-)) - g(u'(t_- - \varepsilon_-)) = \lambda \int_{t_- - \varepsilon_-}^{t_-} (F_n(u, u', u'(t)))(t)\, dt$$
$$= \lambda \int_{t_- - \varepsilon_-}^{t_-} \Big((F(\bar{u}, \widetilde{u'}, L_1))(t) + \frac{p(u'(t))}{n} \Big)\, dt$$
$$\leq \frac{\lambda}{n} \int_{t_- - \varepsilon_-}^{t_-} p(u'(t))\, dt = -\frac{\lambda \varepsilon_-}{n} < 0,$$

which contradicts $g(u'(t_-)) - g(u'(t_- - \varepsilon_-)) = g(L_1) - g(L_1 - \frac{1}{n}) > 0$. Hence $\min\{u'(t);\, t \in J\} > L_1 - \frac{1}{n}$.

Let $A_i = \min\{A_1, A_2\}$. Then

$$|u(t)| = \Big| u(t_i) + \int_{t_i}^{t} u'(s)\, ds \Big| \leq \min\{A_1, A_2\} + T \max\Big\{ -L_1 + \frac{1}{n}, L_3 + \frac{1}{n} \Big\}$$
$$= \min\{A_1, A_2\} + \Big(\max\{-L_1, L_3\} + \frac{1}{n} \Big) T = U + \frac{T}{n}$$

for $t \in J$. Hence the lemma is proved. □

COROLLARY 2.2. (A priori estimates). *Let assumptions* $(H_1) - (H_4)$ *be satisfied. Let* u *be a solution of BVP* $(6_n)_\lambda$, (2) *for some* $n \geq n_0$ *and* $\lambda \in [0,1]$. *Then*

$$\|u\| \leq U + \frac{T}{n}, \quad D - \frac{1}{n} < u'(t) < H + \frac{1}{n}, \qquad t \in J.$$

PROOF. If $L_1 < L_4$, $L_2 < L_3$, the assertion follows from Lemma 2.1. Let $L_1 > L_4$, $L_2 < L_3$. Then by the same procedure as in the proof of Lemma 2.1 we prove

$$\|u\| \leq U + \frac{T}{n}, \quad L_4 - \frac{1}{n} < u'(t) < L_3 + \frac{1}{n}, \qquad t \in J.$$

Similarly for $L_2 > L_3$. □

LEMMA 2.3. *Let assumptions* $(H_1) - (H_4)$ *be satisfied with* $L_1 < L_4$ *and* $L_2 < L_3$. *Then for sufficiently large* $n \in \mathbb{N}$ *BVP* $(6_n)_1$, (2) *has a solution* u *satisfying the inequalities*

$$\|u\| \leq U + \frac{T}{n}, \quad L_1 - \frac{1}{n} < u'(t) < L_3 + \frac{1}{n}, \qquad t \in J.$$

PROOF. Fix $n \in \mathbb{N}$, $n \geq n_0$. Set $K = \max\{-D, H\}$,

$$G(v) = \max\{g(v), -g(-v)\} \quad \text{for } v \in [0, \infty),$$

$$\begin{aligned} \Omega \;=\; & \Big\{(x,y,z,b,c);\, (x,y,z,b,c) \in C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2, \\ & \|x\| < U + (1+\mu)T,\, \|y\| < K+1,\, \|z\| < K+1, \\ & |b| < U + (1+\mu)T,\, |c| < G(K+1)\Big\} \end{aligned}$$

and define the operators

$$Z : \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,$$
$$W : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$$

by

$$Z(x,y,z,b,c) = \Big(b + g^{-1}(c)t,\, g^{-1}(c),\, g^{-1}(c), b - x(\alpha_1(x,y)),\, c - x(\alpha_2(x,y))\Big),$$
$$W(\lambda, x, y, z, b, c) = \lambda Z(x,y,z,b,c).$$

We first prove that

$$D(I - Z, \Omega, 0) \neq 0, \tag{9}$$

where "D" is the Leray-Schauder degree and I is the identical operator on the Banach space $C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$. It is easy to check that W is a compact operator. Assume

$$W(\lambda_0, x_0, y_0, z_0, b_0, c_0) = (x_0, y_0, z_0, b_0, c_0)$$

for some $(\lambda_0, x_0, y_0, z_0, b_0, c_0) \in [0,1] \times \partial\Omega$. Then

$$x_0(t) = \lambda_0(b_0 + g^{-1}(c_0)t), \quad y_0(t) = \lambda_0 g^{-1}(c_0), \quad z_0(t) = \lambda_0 g^{-1}(c_0), \tag{10}$$

$$b_0 = \lambda_0(b_0 - x_0(\alpha_1(x_0, y_0))), \tag{11}$$

$$c_0 = \lambda_0(c_0 - x_0(\alpha_2(x_0, y_0))). \tag{12}$$

From (10)–(12) we deduce that $y_0 = x_0'$,

$$\begin{aligned} b_0 &= \lambda_0\Big(b_0 - \lambda_0 b_0 - \lambda_0 g^{-1}(c_0)\alpha_1(x_0, x_0')\Big), \\ c_0 &= \lambda_0\Big(c_0 - \lambda_0 b_0 - \lambda_0 g^{-1}(c_0)\alpha_2(x_0, x_0')\Big), \end{aligned} \tag{13}$$

and so

$$b_0 = -\frac{\lambda_0^2 g^{-1}(c_0)\alpha_1(x_0, x_0')}{1 - \lambda_0 + \lambda_0^2}, \tag{14}$$

$$(1 - \lambda_0)(b_0 - c_0) = \lambda_0^2 g^{-1}(c_0)(\alpha_2(x_0, x_0') - \alpha_1(x_0, x_0')). \tag{15}$$

If $\lambda_0 = 0$ then $(x_0, y_0, z_0, b_0, c_0) = (0,0,0,0,0)$. Assume $\lambda_0 = 1$. Then (cf. (13)) $b_0 = -g^{-1}(c_0)\alpha_1(x_0, x_0')$, $b_0 = -g^{-1}(c_0)\alpha_2(x_0, x_0')$, and consequently $0 = g^{-1}(c_0)(\alpha_1(x_0, x_0') - \alpha_2(x_0, x_0'))$. Since $\alpha_2(x_0, x_0') - \alpha_1(x_0, x_0') > 0$ by (H_1), $c_0 = 0$ and (10) and (14) show that $(x_0, y_0, z_0, b_0, c_0) = (0,0,0,0,0)$. Let $\lambda_0 \in (0,1)$. Assume $c_0 \neq 0$. Then from (14) and (15) we obtain that

$$-(1-\lambda_0)\Big(\frac{c_0}{g^{-1}(c_0)} + \frac{\lambda_0^2\alpha_1(x_0, x_0')}{1 - \lambda_0 + \lambda_0^2}\Big) = \lambda_0^2(\alpha_2(x_0, x_0') - \alpha_1(x_0, x_0')).$$

Since $-(1-\lambda_0)\Big(\frac{c_0}{g^{-1}(c_0)} + \frac{\lambda_0^2\alpha_1(x_0, x_0')}{1-\lambda_0+\lambda_0^2}\Big) < 0$ and $\lambda_0^2(\alpha_2(x_0, x_0') - \alpha_1(x_0, x_0')) > 0$, we obtain a contradiction. Hence $c_0 = 0$, and so $(x_0, y_0, z_0, b_0, c_0) = (0,0,0,0,0)$.

We have proved $(x_0, y_0, z_0, b_0, c_0) = (0,0,0,0,0) \notin \partial\Omega$, a contradiction. By the theory of homotopy (see e.g. [**D**] and [**M**])

$$\begin{aligned} D(I - Z, \Omega, 0) &= D(I - W(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) \\ &= D(I - W(0, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = D(I, \Omega, 0) = 1, \end{aligned}$$

which proves (9).

Let the operators

$$Z_1 : \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,$$

$$W_1 : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$$

be given by

$$Z_1(x, y, z, b, c) = Z(x, y, z, b, c) + (0, 0, 0, p_1(x, y), p_2(x, y)),$$

$$W_1(\lambda, x, y, z, b, c) = Z(x, y, z, b, c) + \lambda(0, 0, 0, p_1(x, y), p_2(x, y)).$$

Then W_1 is a compact operator and $W_1(1, \cdot, \cdot, \cdot, \cdot, \cdot) = Z_1(\cdot, \cdot, \cdot, \cdot, \cdot)$. Assume

$$W_1(\lambda_1, x_1, y_1, z_1, b_1, c_1) = (x_1, y_1, z_1, b_1, c_1)$$

for a $(\lambda_1, x_1, y_1, z_1, b_1, c_1) \in [0,1] \times \partial\Omega$. Then

$$x_1(t) = b_1 + g^{-1}(c_1)t, \quad y_1(t) = g^{-1}(c_1), \quad z_1(t) = g^{-1}(c_1),$$

$$x_1(\alpha_1(x_1, x_1')) = \lambda_1 p_1(x_1, x_1'), \quad x_1(\alpha_2(x_1, x_1')) = \lambda_1 p_2(x_1, x_1'),$$

and so

$$b_1 + g^{-1}(c_1)\alpha_1(x_1, x_1') = \lambda_1 p_1(x_1, x_1'), \quad b_1 + g^{-1}(c_1)\alpha_2(x_1, x_1') = \lambda_1 p_2(x_1, x_1').$$

Thus (cf. (H_1), (H_2) and (H_4))

$$\begin{aligned} |g^{-1}(c_1)|(\alpha_2(x_1, x_1') - \alpha_1(x_1, x_1')) &= \lambda_1 |p_1(x_1, x_1') - p_2(x_1, x_1')| \\ &\le \mu(\alpha_2(x_1, x_1') - \alpha_1(x_1, x_1')), \end{aligned}$$

which yields $|g^{-1}(c_1)| \le \mu$. Whence

$$|b_1| \le \mu T + \min\{A_1, A_2\} \le U.$$

Consequently,

$$\|x_1\| \le U + \mu T, \quad \|y_1\| \le \mu, \quad \|z_1\| \le \mu, \quad |b_1| \le U, \quad |c_1| \le G(\mu),$$

which contradicts $(x_1, y_1, z_1, b_1, c_1) \in \partial\Omega$. Thus (cf. (9))

$$\begin{aligned} D(I - Z_1, \Omega, 0) &= D(I - W_1(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) \\ &= D(I - W_1(0, , \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = D(I - Z, \Omega, 0) \ne 0. \end{aligned} \tag{16}$$

Finally define

$$\begin{aligned} S &: \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2, \\ V &: [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2 \end{aligned}$$

by

$$\begin{aligned} S(x, y, z, b, c) \;\; = \;\; \Bigg(& b + \int_0^t g^{-1}\Big(c + \int_0^s (F_n(x, y, z(\nu)))(\nu)\, d\nu\Big)\, ds, \\ & g^{-1}\Big(c + \int_0^t (F_n(x, y, z(s)))(s)\, ds\Big), \\ & g^{-1}\Big(c + \int_0^t (F_n(x, y, z(s)))(s)\, ds\Big), \\ & b - x(\alpha_1(x, y)) + p_1(x, y),\; c - x(\alpha_2(x, y)) + p_2(x, y) \Bigg), \end{aligned}$$

$$\begin{aligned} V(\lambda,x,y,z,b,c) \;=\; \Big(& b+\int_0^t g^{-1}\Big(c+\lambda\int_0^s (F_n(x,y,z(\nu)))(\nu)\,d\nu\Big)\,ds, \\ & g^{-1}\Big(c+\lambda\int_0^t (F_n(x,y,z(s)))(s)\,ds\Big), \\ & g^{-1}\Big(c+\lambda\int_0^t (F_n(x,y,z(s)))(s)\,ds\Big), \\ & b-x(\alpha_1(x,y))+p_1(x,y),\ c-x(\alpha_2(x,y))+p_2(x,y)\Big). \end{aligned}$$

Obviously, if (x,y,z,b,c) is a fixed point of the operator S, then x is a solution of BVP $(6_n)_1$, (2) and $x'=y=z$, $b=x(0)$, $c=g(x'(0))$. Conversely, if x is a solution of BVP $(6_n)_1$, (2) and $(x,x',x',x(0),g(x'(0)))\in\bar{\Omega}$, then $(x,x',x',x(0),g(x'(0)))$ is a fixed point of S.

To prove that V is a compact operator, let $\{(\lambda_j,x_j,y_j,z_j,b_j,c_j)\}\subset[0,1]\times\bar{\Omega}$. Set

$$(u_j,v_j,w_j,B_j,C_j)=V(\lambda_j,x_j,y_j,z_j,b_j,c_j),\qquad j\in\mathbb{N}$$

and

$$P(v)=\max\{g^{-1}(v),\,-g^{-1}(-v)\},\quad v\in[0,\infty).$$

Then

$$u_j(t)=b_j+\int_0^t g^{-1}\Big(c_j+\lambda_j\int_0^s (F_n(x_j,y_j,z_j(\nu)))(\nu)\,d\nu\Big)\,ds,$$

$$v_j(t)=w_j(t)=g^{-1}\Big(c_j+\lambda_j\int_0^t (F_n(x_j,y_j,z_j(s)))(s)\,ds\Big)\ (=u_j'(t)),$$

$$B_j=b_j-x_j(\alpha_1(x_j,x_j'))+p_1(x_j,x_j'),\quad C_j=c_j-x_j(\alpha_2(x_j,x_j'))+p_2(x_j,x_j'),$$

and from the property (c) of F and (5), it follows that there exists $k\in L_1(J)$ such that

(17) $\quad |(F_n(x_j,y_j,z_j(t)))(t)|\le k(t)$ for a.e. $t\in J$ and each $j\in\mathbb{N}$.

Consequently,

$$|u_j(t)|\le U+(1+\mu)T+TP\Big(G(K+1)+\int_0^T k(t)\,dt\Big),$$

$$|u_j'(t)|=|v_j(t)|=|w_j(t)|\le P\Big(G(K+1)+\int_0^T k(t)\,dt\Big),$$

$$|g(u_j'(t_1))-g(u_j'(t_2))|=|g(v_j(t_1))-g(v_j(t_2))|\le\Big|\int_{t_1}^{t_2} k(t)\,dt\Big|,$$

$$|B_j|\le 2(U+(1+\mu)T)+A_1,\quad |C_j|\le G(K+1)+U+(1+\mu)T+A_2$$

for t, t_1, $t_2 \in J$ and $j \in \mathbb{N}$. Going if necessary to a subseguence, we can assume, by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem, that $\{(u_j, v_j, w_j, B_j, C_j)\}$ is convergent in $C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$. Since V is continuous (see the property (b) of F and the definition of F_n), the compactness of V is proved.

Assume

$$V(\lambda_0, x_0, y_0, z_0, b_0, c_0) = (x_0, y_0, z_0, b_0, c_0)$$

for some $(\lambda_0, x_0, y_0, z_0, b_0, c_0) \in [0,1] \times \partial\Omega$. Then x_0 is a solution of BVP $(6_n)_{\lambda_0}$, (2) and $x_0' = y_0 = z_0$, $b_0 = x_0(0)$, $c_0 = g(x_0'(0))$. By Lemma 2.1, $\|x_0\| \le U + \frac{T}{n}$, $L_1 - \frac{1}{n} < x_0'(t) < L_3 + \frac{1}{n}$ for $t \in J$, and so

$$\|y_0\| = \|z_0\| < K + 1, \quad |b_0| \le U + \frac{T}{n}, \quad |c_0| < G(K+1),$$

which contradicts $(x_0, y_0, z_0, b_0, c_0) \in \partial\Omega$.

By the theory of homotopy (cf. (16))

$$D(I - S, \Omega, 0) = D(I - V(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0)$$
$$= D(I - V(0, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = D(I - Z_1, \Omega, 0) \ne 0.$$

Consequently, there exists a fixed point $(u, v, z, b, c) \in \Omega$ of the operator S. Then u is a solution of BVP $(6_n)_1$, (2) and Lemma 2.1 shows that $\|u\| \le U + \frac{T}{n}$, $L_1 - \frac{1}{n} < u'(t) < L_3 + \frac{1}{n}$ for $t \in J$. □

COROLLARY 2.4. *Let assumptions $(H_1) - (H_4)$ be satisfied. Then for sufficiently large $n \in \mathbb{N}$ BVP $(6_n)_1$, (2) has a solution u satisfying*

$$\|u\| \le U + \frac{T}{n}, \quad D - \frac{1}{n} < u'(t) < H + \frac{1}{n}, \qquad t \in J.$$

PROOF. If $L_1 < L_4$, $L_2 < L_3$, the assertion follows from Lemma 2.3. Let $L_1 > L_4$, $L_2 < L_3$. By the same arguments as in the proof of Lemma 2.3 we prove that for sufficiently large $n \in \mathbb{N}$ BVP $(6_n)_1$, (2) has a solution u such that

$$\|u\| \le U + \frac{T}{n}, \quad L_4 - \frac{1}{n} < u'(t) < L_3 + \frac{1}{n}, \qquad t \in J.$$

Similarly for $L_2 > L_3$. □

THEOREM 2.5. *Let assumptions $(H_1) - (H_4)$ be satisfied. Then BVP (1), (2) has a solution u and the estimates*

$$\|u\| \le U, \quad D \le u'(t) \le H \tag{18}$$

for $t \in J$ are fulfilled.

PROOF. By Corollary 2.4, BVP $(6_n)_1$, (2) has a solution u_n for sufficiently large $n \in \mathbb{N}$ and

$$\|u_n\| \le U + \frac{T}{n}, \quad D - \frac{1}{n} \le u_n'(t) \le H + \frac{1}{n}, \quad t \in J.$$

Moreover, the property (c) of F implies that there is $k_1 \in L_1(J)$ such that $|(F_n(u_n, u'_n, u'_n(t))(t)| \le k_1(t)$ for a.e. $t \in J$, and so

$$|g(u'_n(t_1)) - g(u'_n(t_2))| \le \Big| \int_{t_1}^{t_2} k_1(t)\, dt \Big|$$

for t_1, $t_2 \in J$ and sufficiently large $n \in \mathbb{N}$. Thus $\{u_n\}$, $\{u'_n\}$ are bounded in $C^0(J)$, $\{u'_n(t)\}$ is equicontinuous on J since g is a continuous and increasing function. By the Arzelà–Ascoli theorem, we can choose a subsequence $\{u_{k_n}\}$ converging (in $C^1(J)$) to u. One can see that u fulfils (2) and (18), and (see the property (b) of F)

$$\begin{aligned}
&\lim_{n\to\infty}(F_{k_n}(u_{k_n}, u'_{k_n}, u'_{k_n}(t)))(t)\\
&= \lim_{n\to\infty}\Big((F(\bar{u}_{k_n}, \widetilde{u'}_{k_n}, [u'_{k_n}(t)]_{k_n}))(t) + \tfrac{p(u'_{k_n}(t))}{k_n}\Big)\\
&= (F(u, u', u'(t)))(t)
\end{aligned}$$

in $L_1(J)$. Thus, u is a solution of BVP (1), (2) satisfying inequalities (18). □

EXAMPLE 2.1. Let $J = [0,3]$ and $h : \mathbb{R} \to \mathbb{R}$, $F_1 : C^0(J) \times C^0(J) \to L_1(J)$ be continuous and $h(L_i) = 0$ $(i = 1, 2, 3, 4)$ where $L_1 < L_4 \le -2$, $2 \le L_2 < L_3$. Consider BVP

(19) $$(g(x'(t)))' = h(x'(t))(F_1(x, x'))(t),$$

(20) $$\begin{aligned}
x\Big(\tfrac{2|x(\xi)|}{1+x^2(\xi)}\Big) &= \sin\Big(\int_0^3 \sqrt{|x(t)| + (x'(t))^2}\, dt\Big),\\
x(3 - |\sin x'(\varepsilon)|) &= \cos x(\nu),
\end{aligned}$$

where $\xi, \varepsilon, \nu \in J$. Applying Theorem 2.5 (with $F(x, y, a) = h(a)F_1(x, y)$, $\mu = 2$, $A_1 = A_2 = 1$, $\alpha_1(x, y) = \frac{2|x(\xi)|}{1+x^2(\xi)}$, $\alpha_2(x, y) = 3 - |\sin y(\varepsilon)|$, $p_1(x, y) = \sin\Big(\int_0^3 \sqrt{|x(t)| + (y(t))^2}\, dt\Big)$, $p_2(x, y) = \cos x(\nu)$,), BVP (19), (20) has a solution u satisfying the inequalities

$$\|u\| \le 1 + 3\max\{-L_1, L_3\}, \quad L_1 \le u'(t) \le L_3$$

for $t \in J$.

3. BVP (1), (3)

In [**RS**] problems for second order functional differential equations with boundary conditions $\alpha(x) = 0$, $x'(1) = 0$ or $\alpha(x) = 0$, $x'(0) = 0$ were also considered. Here $\alpha : C^0([0,1]) \to \mathbb{R}$ is a linear bounded and increasing (i.e. $x, y \in C^0([0,1])$, $x(t) < y(t)$ for $t \in [0,1] \Rightarrow \alpha(x) < \alpha(y)$) functional. We observe that $\alpha(x) = 0$ for an $x \in C^0([0,1])$ implies $x(\xi) = 0$ with a $\xi \in [0,1]$. The authors proved existence results for the above BVPs under assumptions which are of the type of our assumption (H_6) but only with two constants

K_1, K_2 or K_3, K_4. We observe that these assumptions are not sufficient for existence results of BVP (1), (3) as follows from Example 3.1.

EXAMPLE 3.1. Consider the differential equation $x'' = \varepsilon x'^3$ on $J = [0,2]$ with the boundary conditions $x(\frac{3}{4}) = 0$, $x'(1) = 1$. Here $\varepsilon = \pm 1$. This BVP is the special case of BVP (1), (3) (with $F(x,y,a) = \varepsilon a^3$, $\beta_1(x,y) = \frac{3}{4}$, $\beta_2(x,y) = 1$, $r_1(x,y) = 0$, $r_2(x,y) = 1$). Clearly, $(F(x,y,-2))(t) = -8\varepsilon$, $(F(x,y,2))(t) = 8\varepsilon$ for $t \in J$ and $x, y \in C^0(J)$. But our BVPs have no solution since for $\varepsilon = -1$ (resp. $\varepsilon = 1$) the unique solution is defined only on the interval $(\frac{1}{2}, 2]$ (resp. $[0, \frac{3}{2})$).

The proofs of existence results for BVP (1), (3) are very similar to those for BVP (1), (2). Let assumptions (H_5) and (H_6) be satisfied and let $|K_4 - K_1| > \frac{2}{n_0}$, $|K_3 - K_2| > \frac{2}{n_0}$ for an $n_0 \in \mathbb{N}$. Set

$$E_1^* = K_1 + \tfrac{\operatorname{sign}(K_4-K_1)-1}{2}(K_1 - K_4),\ E_2^* = K_2 + \tfrac{\operatorname{sign}(K_3-K_2)-1}{2}(K_2 - K_3),$$

$$E_3^* = K_3 - \tfrac{\operatorname{sign}(K_3-K_2)-1}{2}(K_2 - K_3),\ E_4^* = K_4 - \tfrac{\operatorname{sign}(K_4-K_1)-1}{2}(K_1 - K_4).$$

Then $E_1^* < E_4^* \le -N$, $N \le E_2^* < E_3^*$ and $D_* = E_1^*$, $H_* = E_3^*$. For each $n \ge n_0$, x, $y \in C^0(J)$ and $a \in \mathbb{R}$, define x_*, $\hat{y} \in C^0(J)$ and $\{a\}_n \in \mathbb{R}$ by

$$x_*(t) = \begin{cases} U_* & \text{for } x(t) > U_* \\ x(t) & \text{for } |x(t)| \le U_* \\ -U_* & \text{for } x(t) < -U_*, \end{cases}$$

$$\hat{y}(t) = \begin{cases} H_* & \text{for } y(t) > H_* \\ y(t) & \text{for } D_* \le y(t) \le H_* \\ D_* & \text{for } y(t) < D_*, \end{cases}$$

$$\{a\}_n = \begin{cases} E_3^* & \text{for } a \ge E_3^* \\ a & \text{for } E_2^* + \frac{2}{n} < a < E_3^* \\ -E_2^* + 2a - \frac{2}{n} & \text{for } E_2^* + \frac{1}{n} < a \le E_2^* + \frac{2}{n} \\ E_2^* & \text{for } E_2^* < a \le E_2^* + \frac{1}{n} \\ a & \text{for } E_4^* \le a \le E_2^* \\ E_4^* & \text{for } E_4^* - \frac{1}{n} \le a < E_4^* \\ -E_4^* + 2a + \frac{2}{n} & \text{for } E_4^* - \frac{2}{n} \le a < E_4^* - \frac{1}{n} \\ a & \text{for } E_1^* \le a < E_4^* - \frac{2}{n} \\ E_1^* & \text{for } a < E_1^*. \end{cases}$$

Let $l : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property:

$$|l(v)| \le 1 \quad \text{for } v \in \mathbb{R},$$

$$l(v) = 1 \quad \text{for } v \in [K_4 - \tfrac{1}{n_0}, K_4] \cup [K_2, K_2 + \tfrac{1}{n_0}], \tag{21}$$

$$l(v) = -1 \quad \text{for } v \in [K_1 - \tfrac{1}{n_0}, K_1] \cup [K_3, K_3 + \tfrac{1}{n_0}].$$

Set

$$(F_n^*(x, y, a))(t) = (F(x_*, \hat{y}, \{a\}_n))(t) + \frac{l(a)}{n}$$

for $(x, y, a) \in C^0(J) \times C^0(J) \times \mathbb{R}$ and $n \in \mathbb{N}$, $n \ge n_0$.

Consider the two–parameter family of the functional differential equations

$$(g(x'(t)))' = \lambda(F_n^*(x, x', x'(t)))(t), \qquad \lambda \in [0,1],\ n \ge n_0. \tag{22_n)$_\lambda$}$$

LEMMA 3.1. (A priori estimates). *Let assumptions* (H_5) *and* (H_6) *be satisfied with* $K_1 < K_4$ *and* $K_2 < K_3$ *and let BVP* $(22_n)_\lambda, (3)$ *has a solution* u *for some* $\lambda \in [0,1]$ *and* $n \ge n_0$. *Then*

$$\|u\| \le U_* + \frac{T}{n}, \quad K_1 - \frac{1}{n} < u'(t) < K_3 + \frac{1}{n}$$

for $t \in J$.

PROOF. Set $T_1 = \beta_1(u, u')$, $T_2 = \beta_2(u, u')$. By (H_5),

$$|u'(T_2)| \le N. \tag{23}$$

If $\lambda = 0$, $g(u'(t)) \equiv$ *const.*; hence (cf. (23)) $|u'(t)| \le N$ for $t \in J$. Let $u'(\xi) = \max\{u'(t); t \in J\} \ge K_3 + \frac{1}{n}$ with a $\xi \in J$. If $\xi \in (T_2, T]$, then there exist $t_0 \in (T_2, \xi)$ and $\varepsilon_0 > 0$ such that $u'(t_0) = K_3$, $u'(t_0 + \varepsilon_0) = K_3 + \frac{1}{n}$ and $K_3 \le u'(t) \le K_3 + \frac{1}{n}$ for $t \in [t_0, t_0 + \varepsilon_0]$. Integrating the equality

$$(g(u'(t))' = \lambda(F_n^*(u, u', u'(t)))(t)$$

for a.e. $t \in J$ from t_0 to $t_0 + \varepsilon_0$ we obtain

$$g(u'(t_0 + \varepsilon_0)) - g(u'(t_0)) = \lambda \int_{t_0}^{t_0+\varepsilon_0} (F_n^*(u, u', u'(t)))(t)\, dt$$

$$= \lambda \int_{t_0}^{t_0+\varepsilon_0} \left((F(u_*, \widehat{u'}, K_3))(t) + \frac{l(u'(t))}{n} \right) dt$$

$$\le \frac{\lambda}{n} \int_{t_0}^{t_0+\varepsilon_0} l(u'(t))\, dt = -\frac{\lambda\varepsilon_0}{n} < 0,$$

which contradicts $g(u'(t_0 + \varepsilon_0)) - g(u'(t_0)) = g(K_3 + \frac{1}{n}) - g(K_3) > 0$. The next part of the proof of the inequalities $K_1 - \frac{1}{n} < u'(t) < K_3 + \frac{1}{n}$, $t \in J$, is similar to that of Lemma 2.1 (with $L_i = K_i$ $(i = 1, 2, 3, 4)$ and $\mu = N$) and therefore it is omitted.

Since $|u(t)| = \left|u(T_1) + \int_{T_1}^t u'(s)\,ds\right| \le |r_1(u,u')| + \left|\int_{T_1}^t |u'(s)|\,ds\right| \le M + T\max\{-K_1 + \frac{1}{n}, K_3 + \frac{1}{n}\} = M + (\max\{-K_1, K_3\} + \frac{1}{n})T$ for $t \in J$, we have $\|u\| \le U_* + \frac{T}{n}$. □

From Lemma 3.1 and applying the same procedure as in the proof of Corollary 2.2 we obtain the following corollary.

COROLLARY 3.2. (A priori estimates). *Let assumptions* (H_5) *and* (H_6) *be satisfied. Let* u *be a solution of BVP* $(22_n)_\lambda$, (3) *for some* $n \ge n_0$ *and* $\lambda \in [0,1]$. *Then*

$$\|u\| \le U_* + \frac{T}{n}, \quad D_* - \frac{1}{n} < u'(t) < H_* + \frac{1}{n}$$

for $t \in J$.

LEMMA 3.3. *Let assumptions* (H_5) *and* (H_6) *be satisfied with* $K_1 < K_4$ *and* $K_2 < K_3$. *Then for sufficiently large* $n \in \mathbb{N}$ *BVP* $(22_n)_1$, (3) *has a solution* u *satisfying*

$$\|u\| \le U_* + \frac{T}{n}, \quad K_1 - \frac{1}{n} < u'(t) < K_3 + \frac{1}{n}, \qquad t \in J.$$

PROOF. Fix $n \in \mathbb{N}$, $n \ge n_0$. Set $K_* = \max\{-D_*, H_*\}$,

$$G(v) = \max\{g(v), -g(-v)\}, \quad P(v) = \max\{g^{-1}(v), -g^{-1}(-v)\}, \quad v \in [0,\infty),$$

$$\begin{aligned}\Omega \;=\; & \Big\{(x,y,z,b,c);\ (x,y,z,b,c) \in C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,\\ & \|x\| < 2U_* + 1,\ \|y\| < K_* + 1,\ \|z\| < K_* + 1,\\ & |b| < 2U_* + 1,\ |c| < G(K_* + 1)\Big\}\end{aligned}$$

and define the operators

$$Z_* : \bar\Omega \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,$$
$$W_* : [0,1] \times \bar\Omega \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$$

by

$$Z_*(x,y,z,b,c) = \Big(b + g^{-1}(c)t,\ g^{-1}(c),\ g^{-1}(c), b - x(\beta_1(x,y)),\ c - x'(\beta_2(x,y))\Big),$$

$$W_*(\lambda,x,y,z,b,c) = \lambda Z_*(x,y,z,b,c).$$

It can be shown without difficulties that W_* is a compact operator. Assume

$$W_*(\lambda_0,x_0,y_0,z_0,b_0,c_0) = (x_0,y_0,z_0,b_0,c_0)$$

for some $(\lambda_0,x_0,y_0,z_0,b_0,c_0) \in [0,1] \times \partial\Omega$. Then

$$x_0(t) = \lambda_0(b_0 + g^{-1}(c_0)t), \quad x_0'(t) = y_0(t) = z_0(t) = \lambda_0 g^{-1}(c_0),$$
$$b_0 = \lambda_0(b_0 - x_0(\beta_1(x_0,x_0'))), \quad c_0 = \lambda_0(c_0 - x_0'(\beta_2(x_0,x_0'))),$$

and so

$$b_0 = \lambda_0\Big(b_0 - \lambda_0 b_0 - \lambda_0 g^{-1}(c_0)\beta_1(x_0, x_0')\Big), \tag{24}$$
$$c_0 = \lambda_0(c_0 - \lambda_0 g^{-1}(c_0)).$$

Thus

$$(1-\lambda_0)c_0 = -\lambda_0^2 g^{-1}(c_0).$$

If $\lambda_0 \in \{0, 1\}$ then $c_0 = 0$. Assume $\lambda_0 \in (0,1)$. If $c_0 \neq 0$ then $\frac{c_0}{g^{-1}(c_0)} = -\frac{\lambda_0^2}{1-\lambda_0}$, which contradicts $\frac{c_0}{g^{-1}(c_0)} > 0$, $-\frac{\lambda_0^2}{1-\lambda_0} < 0$. Hence $c_0 = 0$, and consequently (cf. (24)) $b_0(1-\lambda_0+\lambda_0^2) = 0$ which gives $b_0 = 0$ since $1-\lambda_0+\lambda_0^2 > 0$. We have proved: $(x_0, y_0, z_0, b_0, c_0) = (0,0,0,0,0)$, a contradiction. By the theory of homotopy

$$\begin{aligned} D(I - Z_*, \Omega, 0) &= D(I - W_*(1,\cdot,\cdot,\cdot,\cdot,\cdot), \Omega, 0) \\ &= D(I - W_*(0,\cdot,\cdot,\cdot,\cdot,\cdot), \Omega, 0) = D(I, \Omega, 0) = 1. \end{aligned} \tag{25}$$

Let the operators

$$Z_{*1} : \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,$$
$$W_{*1} : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$$

be given by

$$Z_{*1}(x,y,z,b,c) = Z_*(x,y,z,b,c) + (0,0,0,r_1(x,y), r_2(x,y)),$$
$$W_{*1}(\lambda,x,y,z,b,c) = Z_*(x,y,z,b,c) + \lambda(0,0,0,r_1(x,y), r_2(x,y)).$$

Then W_{*1} is a compact operator. Assume

$$W_{*1}(\lambda_1, x_1, y_1, z_1, b_1, c_1) = (x_1, y_1, z_1, b_1, c_1)$$

for some $(\lambda_1, x_1, y_1, z_1, b_1, c_1) \in [0,1] \times \partial\Omega$. Then

$$x_1(t) = b_1 + g^{-1}(c_1)t, \quad x_1'(t) = y_1(t) = z_1(t) = g^{-1}(c_1), \tag{26}$$
$$x_1(\beta_1(x_1, x_1')) = \lambda_1 r_1(x_1, x_1'), \quad x_1'(\beta_2(x_1, x_1')) = \lambda_1 r_2(x_1, x_1'),$$

and so

$$b_1 + g^{-1}(c_1)\beta_1(x_1, x_1') = \lambda_1 r_1(x_1, x_1'), \tag{27}$$

$$g^{-1}(c_1) = \lambda_1 r_2(x_1, x_1'). \tag{28}$$

From (28) and (H_5) we obtain (see the definition of the function G)

$$|c_1| \leq G(N)$$

and then (cf. (27), (28) and (H_5))

$$|b_1| \leq |g^{-1}(c_1)|T + M \leq |r_2(x_1, x_1')|T + M \leq NT + M,$$

and consequently (cf. (26), (28) and (H_5))

$$\|x_1\| \leq 2NT + M, \quad \|x_1'\| = \|y_1\| = \|z_1\| \leq N.$$

We see that $(x_1, y_1, z_1, b_1, c_1) \notin \partial\Omega$, a contradiction. Thus (cf. (25))

$$
\begin{aligned}
&D(I - Z_{*1}, \Omega, 0) = D(I - W_{*1}(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) \\
(29)\qquad &= D(I - W_{*1}(0, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = D(I - Z_*, \Omega, 0) = 1.
\end{aligned}
$$

Finally define

$$
S_* : \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,
$$

$$
V_* : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2
$$

by the formulas

$$
\begin{aligned}
S_*(x, y, z, b, c) \quad = \quad &\Big(b + \int_0^t g^{-1}\Big(c + \int_0^s (F_n^*(x, y, z(\nu)))(\nu)\, d\nu\Big)\, ds, \\
&g^{-1}\Big(c + \int_0^t (F_n^*(x, y, z(s)))(s)\, ds\Big), \\
&g^{-1}\Big(c + \int_0^t (F_n^*(x, y, z(s)))(s)\, ds\Big), \\
&b - x(\beta_1(x, y)) + r_1(x, y),\ c - y(\beta_2(x, y)) + r_2(x, y)\Big),
\end{aligned}
$$

$$
\begin{aligned}
V_*(\lambda, x, y, z, b, c) \quad = \quad &\Big(b + \int_0^t g^{-1}\Big(c + \lambda \int_0^s (F_n^*(x, y, z(\nu)))(\nu)\, d\nu\Big)\, ds, \\
&g^{-1}\Big(c + \lambda \int_0^t (F_n^*(x, y, z(s)))(s)\, ds\Big), \\
&g^{-1}\Big(c + \lambda \int_0^t (F_n^*(x, y, z(s)))(s)\, ds\Big), \\
&b - x(\beta_1(x, y)) + r_1(x, y),\ c - y(\beta_2(x, y)) + r_2(x, y)\Big).
\end{aligned}
$$

If (x, y, z, b, c) is a fixed point of the operator S_* we can easy verify that x is a solution of BVP $(22_n)_1$, (3) and $x' = y = z$, $b = x(0)$, $c = g(x'(0))$. Conversely, if x is a solution of BVP $(22_n)_1$, (3) and $(x, x', x', x(0), g(x'(0))) \in \bar{\Omega}$, then $(x, x', x', x(0), g(x'(0)))$ is a fixed point of S_*.

Thus to prove our lemma it is sufficient to show that there exists a fixed point of S_*. We now verify that V_* is a compact operator. Let $\{(\lambda_i, x_i, y_i, z_i, b_i, c_i)\} \subset [0,1] \times \bar{\Omega}$ be a sequence and set

$$
(u_i, v_i, w_i, B_i, C_i) = V_*(\lambda_i, x_i, y_i, z_i, b_i, c_i), \quad i \in \mathbb{N}.
$$

Then

$$u_i(t) = b_i + \int_0^t g^{-1}\Big(c_i + \lambda_i \int_0^s (F_n^*(x_i, y_i, z_i(\nu)))(\nu)\, d\nu\Big)\, ds,$$

$$u_i'(t) = v_i(t) = w_i(t) = g^{-1}\Big(c_i + \lambda_i \int_0^t (F_n^*(x_i, y_i, z_i(s)))(s)\, ds\Big)$$

$$B_i = b_i - x_i(\beta_1(x_i, y_i)) + r_1(x_i, y_i), \quad C_i = c_i - y_i(\beta_2(x_i, y_i)) + r_2(x_i, y_i),$$

and from the properties of F it follows the existence of a $q \in L_1(J)$ such that

$$|(F_n^*(x_i, y_i, z_i(t)))(t)| \le q(t) \quad \text{for a.e. } t \in J \text{ and each } i \in \mathbb{N}.$$

Hence

$$|u_i(t)| \le 2U_* + 1 + TP\Big(G(K_* + 1) + \int_0^T q(t)\, dt\Big),$$

$$|u_i'(t)| = |v_i(t)| = |w_i(t)| \le P\Big(G(K_* + 1) + \int_0^T q(t)\, dt\Big),$$

$$|g(u_i'(t_1)) - g(u_i'(t_2))| \le \Big|\int_{t_1}^{t_2} q(t)\, dt\Big|,$$

$$|B_i| \le 4U_* + M + 2, \qquad |C_i| \le G(K_* + 1) + K_* + N + 1$$

for t, t_1, $t_2 \in J$. By the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem, we can select a subsequence $\{(u_{i_n}, v_{i_n}, w_{i_n}, B_{i_n}, C_{i_n})\}$ converging in $C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$. From this and from the continuity of V_* we deduce that V_* is a compact operator.

Assume

$$V_*(\lambda_+, x_+, y_+, z_+, b_+, c_+) = (x_+, y_+, z_+, b_+, c_+)$$

for some $(\lambda_+, x_+, y_+, z_+, b_+, c_+) \in [0,1] \times \partial\Omega$. Then x_+ is a solution of BVP $(22_n)_{\lambda_+}$, (3) and $x_+' = y_+ = z_+$, $b_+ = x_+(0)$, $c_+ = g^{-1}(x_+'(0))$. By Lemma 3.1, $\|x_+\| \le U_* + \frac{T}{n}$, $K_1 - \frac{1}{n} < x_+'(t) < K_3 + \frac{1}{n}$ for $t \in J$, which yields

$$\|y_+\| = \|z_+\| < K_* + 1, \quad |b_+| \le U_* + \frac{T}{n}, \quad |c_+| < G(K_* + 1),$$

contrary to $(x_+, y_+, z_+, b_+, c_+) \in \partial\Omega$.

Hence (cf. (29))

$$D(I - S_*, \Omega, 0) = D(I - V_*(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0)$$

$$= D(I - V_*(0, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = D(I - Z_{*1}, \Omega, 0) \ne 0,$$

and so there exists a fixed point $(u, v, w, b, c) \in \Omega$ of S_*. Then u is a solution of BVP $(22_n)_1$, (3) and Lemma 3.1 shows that $\|u\| \le U_* + \frac{T}{n}$, $K_1 - \frac{1}{n} < u'(t) < K_3 + \frac{1}{n}$ for $t \in J$. □

COROLLARY 3.4. *Let assumptions* (H_5) *and* (H_6) *be satisfied. Then for sufficiently large* $n \in \mathbb{N}$ *BVP* $(22_n)_1$, (3) *has a solution* u *satisfying*

$$\|u\| \le U_* + \frac{T}{n}, \quad D_* - \frac{1}{n} < u'(t) < H_* + \frac{1}{n}, \qquad t \in J.$$

PROOF. If $K_1 < K_4$, $K_2 < K_3$, the assertion follows from Lemma 3.3. If $K_1 > K_4$, $K_2 < K_3$ then replacing K_1 and K_4 and using the same procedure as in the proof of Lemma 3.3 we prove that for $n \in \mathbb{N}$ sufficiently large BVP $(22_n)_1$, (3) has a solution u satisfying

$$\|u\| \le U_* + \frac{T}{n}, \quad K_4 - \frac{1}{n} < u'(t) < K_3 + \frac{1}{n}, \qquad t \in J.$$

Similarly for $K_2 > K_3$. □

THEOREM 3.5. *Let assumptions* (H_5) *and* (H_6) *be satisfied. Then BVP* (1), (3) *has a solution* u *satisfying the inequalities*

$$\|u\| \le U_*, \quad D_* \le u'(t) \le H_* \tag{30}$$

for $t \in J$.

PROOF. By Corollary 3.4, BVP $(22_n)_1$, (3) has a solution u_n for sufficiently large $n \in \mathbb{N}$ and

$$\|u_n\| \le U_* + \frac{T}{n}, \quad D_* - \frac{1}{n} \le u_n'(t) \le H_* + \frac{1}{n}, \qquad t \in J.$$

Moreover (cf. the property (c) of F), there exists $q_1 \in L_1(J)$ such that

$$|(F_n^*(u_n, u_n', u_n'(t)))(t)| \le g_1(t)$$

for a.e. $t \in J$, and so

$$|g(u_n'(t_1)) - g(u_n'(t_2))| \le \left| \int_{t_1}^{t_2} q_1(t)\, dt \right|$$

for $t_1, t_2 \in J$ and sufficiently large $n \in \mathbb{N}$. Thus $\{u_n\}$, $\{u_n'\}$ are bounded in $C^0(J)$, $\{u_n'(t)\}$ is equicontinuous on J. By the Arzelà–Ascoli theorem, we can assume without loss of generality that $\{u_n\}$ is a convergent sequence in $C^1(J)$ and let $\lim_{n\to\infty} u_n = u$. Then u fulfils (3) and (30). Since

$$\lim_{n\to\infty} (F_n^*(u_n, u_n', u_n'(t)))(t) = \lim_{n\to\infty} \Big((F(u_{n*}, \widehat{u_n'}, \{u_n'(t)\}_n))(t) + \frac{l(u_n'(t))}{n} \Big)$$
$$= (F(u, u', u'(t)))(t)$$

in $L_1(J)$, we see that u is a solution of BVP (1), (3). □

EXAMPLE 3.2. Let $J = [0,1]$ and $h : \mathbb{R} \to \mathbb{R}$, $F_1 : C^0(J) \times C^0(J) \to L_1(J)$ be continuous and $h(K_i) = 0$ $(i = 1, 2, 3, 4)$ where $K_1 < K_4 \le -2$, $2 \le K_2 < K_3$. Consider equation (19) and the boundary conditions

$$
\begin{aligned}
x(|\sin(x(\xi)x'(\mu))|) &= \min\{S, \|x\|, \|x'\|\}, \\
x'(|\cos(\|x\| + \|x'\|)|) &= \tfrac{1}{1+x^2(\nu)},
\end{aligned}
\tag{31}
$$

where $S \in \mathbb{R}$ and $\xi, \mu, \nu \in J$. By Theorem 3.5 $\Big($with $F(x,y,a) = h(a)F_1(x,y)$, $\beta_1(x,y) = |\sin(x(\xi)y(\mu))|$, $\beta_2(x,y) = |\cos(\|x\|+\|y\|)|$, $r_1(x,y) = \min\{S, \|x\|,$
$\|y\|\}$, $r_2(x,y) = \frac{1}{1+x^2(\nu)}$, $M = |S|$, $N = 1$, $D_* = K_1$, $H_* = K_3$ and $U_* = |S| + \max\{-K_1, K_3\}\Big)$, BVP (19), (31) has a solution u and

$$\|u\| \le |S| + \max\{-K_1, K_3\}, \quad K_1 \le u'(t) \le K_3, \qquad t \in J.$$

REFERENCES

[B] Brykalov S. A., *Problems for differential equations with monotone boundary conditions.* Diff. Urav. 32(1996), 1322–1330 (in Russian).

[D] Deimling K., *Nonlinear Functional Analysis.* Springer Berlin Heidelberg, 1985.

[K] Kelevedjiev P., *Existence of solutions for two-point boundary value problems.* Nonlinear Analysis 22(1994), 217–244.

[M] Mawhin J., *Topological Degree Methods in Nonlinear Boundary Value Problems.* AMS, Providence, R.I., 1979.

[R] Ruyun Ma, *Existence theorems for a second order m-point boundary value problem.* J. Math. Anal. Appl. 211(1997), 545–555.

[RS] Rachunková I. and Staněk S., *Topological degree methods in functional boundary value problems.* Nonlin. Anal. 27(1996), 153–166.

[RT] Rodriguez A. and Tineo A., *Existence theorems for the Dirichlet problem without growth restrictions.* J. Math. Anal. Appl. 135(1988), 1–7.

[S] Staněk S., *Existence results for functional boundary value problems at resonance.* Math. Slovaca 48(1998), 43–55.

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