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SOLUTION OF THE ULAM STABILITY PROBLEM FOR QUARTIC MAPPINGS

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Abstract. In 1940 S. M. Ulam proposed at the University of Wisconsin the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist." In 1968 S. M. Ulam proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In 1978 P. M. Gruber proposed the Ulam type problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?" According to P. M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-1998 we solved the above Ulam problem, or equivalently the Ulam type problem for linear mappings and also established analogous stability problems for quadratic and cubic mappings. In this paper we introduce the new quartic mappings $F : X \to Y$, satisfying the new quartic functional equation

$$F(x_1 + 2x_2) + F(x_1 - 2x_2) + 6F(x_1) = 4\left[F(x_1 + x_2) + F(x_1 - x_2) + 6F(x_2)\right]$$

for all 2-dimensional vectors $(x_1, x_2) \in X^2$, with X a linear space (Y: = a real complete linear space), and then solve the Ulam stability problem for the above mappings F.

1. Quartic functional equation

Definition 1.1. *Let X be a linear space and let Y be a real complete linear space. Then a mapping $F : X \to Y$, is called quartic, if the new quartic*

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functional equation

$$F(x_1 + 2x_2) + F(x_1 - 2x_2) + 6F(x_1) = 4\left[F(x_1 + x_2) + F(x_1 - x_2) + 6F(x_2)\right] \tag{1}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$ ([14-25]).

Note that mapping F is called *quartic* because the following new algebraic identity

$$(x_1 + 2x_2)^4 + (x_1 - 2x_2)^4 + 6x_1^4 = 4\left[(x_1 + x_2)^4 + (x_1 - x_2)^4 + 6x_2^4\right]$$

holds for all $(x_1, x_2) \in R^2$ and because the functional equation

$$F(2^n x) = (2^n)^4 F(x), \tag{2}$$

holds for all $x \in X$, and all $n \in N$. In fact, substitution of $x_1 = x_2 = 0$ in Equ. 1 yields that

$$F(0) = 0. \tag{3}$$

LEMMA 1.2. *Let $F : X \to Y$ be a quartic mapping satisfying Equ. 1. Then F is an even mapping ; that is, equation*

$$F(-x) = F(x) \tag{4}$$

holds for all $x \in X$.

PROOF. Substituting $x_1 = 0$, $x_2 = -x$ in Equ. 1 and employing 3 one gets that equation

$$F(-2x) + F(2x) + 6F(0) = 4[F(-x) + F(x) + 6F(-x)],$$

or

$$F(-2x) + F(2x) - 4F(x) - 28F(-x) = 0 \tag{5}$$

holds for all $x \in X$. Similarly substituting $x_1 = 0$, $x_2 = x$ in Equ. 1 and employing 3 one gets that equation

$$F(2x) + F(-2x) - 4F(-x) - 28F(x) = 0, \tag{6}$$

holds for all $x \in X$. Functional equations 5 - 6 yield

$$24F(-x) = 24F(x)$$

and thus the required equation 4, completing the proof of Lemma 1.2.□

LEMMA 1.3. *Let $F : X \to Y$ be a quartic mapping satisfying Equ. 1. Then F satisfies the general functional equation*

$$F(x) = 2^{-4n} F(2^n x) \tag{7}$$

for all $x \in X$ and all $n \in N$.

PROOF. Employing Eqs. 4-5 one gets the basic equation

$$2F(2x) - 32F(x) = 0,$$

or

$$F(x) = 2^{-4} F(2x) \tag{8}$$

for all $x \in X$.

Then induction on $n \in N$ with $x \to 2^{n-1}x$ in the basic equation 8 yields Equ. 7. In fact, the basic equation 8 with $x \to 2^{n-1}x$ yields that the functional equation

$$F(2^{n-1}x) = 2^{-4} F(2^n x) \tag{9}$$

holds for all $x \in X$.

Moreover by induction hypothesis with $n \to n-1$ in the general equation 7 one gets that

$$F(x) = 2^{-4(n-1)} F(2^{n-1}x) \tag{10}$$

holds for all $x \in X$. Thus functional equations 9-10 imply

$$F(x) = 2^{-4(n-1)} 2^{-4} F(2^n x),$$

or

$$F(x) = 2^{-4n} F(2^n x),$$

for all $x \in X$ and all $n \in N$, completing the proof of the required general functional equation 7 and hence the proof of Lemma 1.3.□

2. QUARTIC FUNCTIONAL INEQUALITY

DEFINITION 2.1. *Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $f : X \to Y$, is called approximately quartic if the new quartic functional inequality*

$$\begin{aligned} &\|f(x_1 + 2x_2) + f(x_1 - 2x_2) + 6f(x_1) - \\ &4\,[f(x_1 + x_2) + f(x_1 - x_2) + 6f(x_2)]\,\| \le c \end{aligned} \tag{11}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$ with a constant $c \ge 0$ (independent of x_1, x_2).

DEFINITION 2.2. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that there exists a constant $c \geq 0$ (independent of $x \in X$). Then a quartic mapping $F : X \to Y$, is said that exists near an approximately quartic mapping $f : X \to Y$ if the following inequality*

$$\|f(x) - F(x)\| \leq \frac{17}{180}c \tag{12}$$

holds for all $x \in X$.

THEOREM 2.3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition the above-mentioned mappings F, f and the three definitions. Then the limit*

$$F(x) = \lim_{n\to\infty} 2^{-4n} f(2^n x). \tag{13}$$

exists for all $x \in X$ and all $n \in N$ and $F : X \to Y$ is the unique quartic mapping near the approximately quartic mapping $f : X \to Y$.

PROOF. (of Existence in Theorem)
Substitution of $x_1 = x_2 = 0$ in Inequ. 11 yields that

$$\|8f(0) - 4\,[8f(0)]\,\| \leq c,$$

or

$$\|f(0)\| \leq \frac{c}{24}\left(= \frac{c}{4!}\right). \tag{14}$$

LEMMA 2.4. *Let $f : X \to Y$ be an approximately quartic mapping satisfying Inequ. 11. Then f is an approximately even mapping ; that is, inequality*

$$\|f(-x) - f(x)\| \leq \frac{c}{12}\left(= \frac{2c}{4!}\right) \tag{15}$$

holds for all $x \in X$ with constant $c \geq 0$ (independent of $x \in X$).

PROOF. Substituting $x_1 = 0$, $x_2 = -x$ in Inequ. 11 one gets that inequality

$$\|f(-2x) + f(2x) + 6f(0) - 4\,[f(-x) + f(x) + 6f(-x)]\,\| \leq c,$$

or

$$\|f(-2x) + f(2x) - 28f(-x) - 4f(x) + 6f(0)\| \leq c \tag{16}$$

holds for all $x \in X$. Similarly substituting $x_1 = 0$, $x_2 = x$ in Inequ. 11 one gets that inequality

$$\|f(-2x) + f(2x) - 28f(x) - 4f(-x) + 6f(0)\| \leq c \tag{17}$$

holds for all $x \in X$.

Functional inequalities 16- 17 and triangle inequality yield

$$\begin{aligned} 24\|f(-x)-f(x)\| &= \|-24\,[f(-x)-f(x)]\,\| \\ &= \|\,[f(-2x)+f(2x)-28f(-x)-4f(x)+6f(0)] \\ &\quad -[f(-2x)+f(2x)-4f(-x)-28f(x)+6f(0)]\,\| \\ &\leq \|f(-2x)+f(2x)-28f(-x)-4f(x)+6f(0)\| \\ &\quad +\|f(-2x)+f(2x)-4f(-x)-28f(x)+6f(0)\| \\ &\leq c+c=2c, \end{aligned}$$

or

$$\|f(-x)-f(x)\| \leq \frac{c}{12},$$

completing the proof of Lemma 2.4. □

LEMMA 2.5. *Let $f: X \to Y$ be an approximately quartic mapping satisfying Inequ. 11. Then f satisfies the general functional inequality*

$$\|f(x)-2^{-4n}f(2^n x)\| \leq \frac{17}{180}(1-2^{-4n})c, \tag{18}$$

for all $x \in X$ and all $n \in N$ with constant $c \geq 0$ (independent of $x \in X$).

PROOF. Employing Ineqs. 17 and triangle inequality one yields inequality

$$\begin{aligned} &\|f(-2x)+f(2x)-4f(-x)-28f(x)\| \\ &\quad = \|\,[f(-2x)+f(2x)-4f(-x)-28f(x)+6f(0)]-[6f(0)]\,\| \\ &\quad \leq \|f(-2x)+f(2x)-4f(-x)-28f(x)+6f(0)\|+\|6f(0)\| \\ &\quad \leq c+6\left(\frac{c}{24}\right)=c+\frac{c}{4}=\frac{5}{4}c, \end{aligned}$$

or

$$\|f(-2x)+f(2x)-4f(-x)-28f(x)\| \leq \frac{5}{4}c, \tag{19}$$

for all $x \in X$.

Applying Ineqs. 14 – 15 – 17 of Lemma 2.4 and triangle inequality one gets that the basic inequality

$$\begin{aligned} &2\|f(2x)-16f(x)\| = \|\,[f(-2x)+f(2x)-4f(-x)-28f(x)] \\ &-[f(-2x)-f(2x)]+4\,[f(-x)-f(x)]\,\| \\ &\leq \|f(-2x)+f(2x)-4f(-x)-28f(x)\|+ \\ &\|f(-2x)-f(2x)\|+4\|f(-x)-f(x)\| \\ &\leq \tfrac{5}{4}c+\tfrac{1}{12}c+\tfrac{1}{12}c=\tfrac{17}{6}c, \end{aligned}$$

or

$$\|f(x)-2^{-4}f(2x)\| \leq \frac{17}{192}c,$$

or

$$\|f(x)-2^{-4}f(2x)\| \leq \frac{17}{180}(1-2^{-4})c \tag{20}$$

holds for all $x \in X$ with constant $c \geq 0$ (independent of $x \in X$).

Replacing now x with $2x$ in basic Inequ. 20 one concludes that

$$\|f(2x) - 2^{-4} f(2^2 x)\| \leq \frac{17}{180}(1 - 2^{-4})c$$

or

$$\|2^{-4} f(2x) - 2^{-8} f(2^2 x)\| \leq \frac{17}{180}(2^{-4} - 2^{-8})c \tag{21}$$

holds for all $x \in X$.

Functional inequalities 20 - 21 and the triangle inequality yield

$$\begin{aligned} &\|f(x) - 2^{-8} f(2^2 x)\| \leq \|f(x) - 2^{-4} f(2x)\| \\ &+\|2^{-4} f(2x) - 2^{-8} f(2^2 x)\| \\ &\leq \tfrac{17}{180}\left[(1 - 2^{-4}) + (2^{-4} - 2^{-8})\right] c \end{aligned}$$

or that the functional inequality

$$\|f(x) - 2^{-8} f(2^2 x)\| \leq \frac{17}{180}(1 - 2^{-8})c, \tag{22}$$

holds for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to 2^{n-1}x$ in the basic Inequ. 20 claim that the general functional inequality 18 holds for all $x \in X$ and all $n \in N$ with constant $c \geq 0$ (independent of $x \in X$).

In fact, the basic inequality 20 with $x \to 2^{n-1}x$ yield the functional inequality

$$\|f(2^{n-1}x) - 2^{-4} f(2^n x)\| \leq \frac{17}{180}(1 - 2^{-4})c,$$

or that the functional inequality

$$\|2^{-4(n-1)} f(2^{n-1}x) - 2^{-4n} f(2^n x)\| \leq \frac{17}{180}(2^{-4(n-1)} - 2^{-4n})c, \tag{23}$$

holds for all $x \in X$.

Moreover, by induction hypothesis with $n \to n-1$ in the general inequality 18 one gets that

$$\|f(x) - 2^{-4(n-1)} f(2^{n-1}x)\| \leq \frac{17}{180}(1 - 2^{-4(n-1)})c \tag{24}$$

holds for all $x \in X$.

Thus functional inequalities 23 - 24 and the triangle inequality imply

$$\begin{aligned} &\|f(x) - 2^{-4n} f(2^n x)\| \leq \|f(x) - 2^{-4(n-1)} f(2^{(n-1)}x)\| \\ &+\|2^{-4(n-1)} f(2^{n-1}x) - 2^{-4n} f(2^n x)\| \\ &\leq \tfrac{17}{180}\left[(1 - 2^{-4(n-1)}) + (2^{-4(n-1)} - 2^{-4n})\right] c, \end{aligned}$$

or

$$\|f(x) - 2^{-4n} f(2^n x)\| \leq \frac{17}{180}(1 - 2^{-4n})c,$$

completing the proof of the required general functional inequality 18, and thus the proof of Lemma 2.5. □

LEMMA 2.6. *Let $f : X \to Y$ be an approximately quartic mapping satisfying Inequ. 11. Then the sequence*

$$\{2^{-4n} f(2^n x)\} \tag{25}$$

converges.

PROOF. Note that from the general functional inequality 18 and the completeness of Y, one proves that the above-mentioned sequence 25 is a Cauchy sequence. In fact, if $i > j > 0$, then

$$\|2^{-4i} f(2^i x) - 2^{-4j} f(2^j x)\| = 2^{-4j} \|2^{-4(i-j)} f(2^i x) - f(2^j x)\|, \tag{26}$$

holds for all $x \in X$, and all $i, j \in N$.

Setting $h = 2^j x$ in 26 and employing the general functional inequality 18 one concludes that

$$\begin{aligned}
&\|2^{-4i} f(2^i x) - 2^{-4j} f(2^j x)\| = \\
&2^{-4j} \|2^{-4(i-j)} f(2^{i-j} h) - f(h)\| \\
&\leq 2^{-4j} \tfrac{17}{180} (1 - 2^{-4(i-j)}) c,
\end{aligned}$$

or

$$\|2^{-4i} f(2^i x) - 2^{-4j} f(2^j x)\| \leq \frac{17}{180} (2^{-4j} - 2^{-4i}) c < \frac{17}{180} 2^{-4j} c$$

or

$$\lim_{j \to \infty} \|2^{-4i} f(2^i x) - 2^{-4j} f(2^j x)\| = 0, \tag{27}$$

which yields that the sequence 25 is a Cauchy sequence, and thus the proof of Lemma 2.6 is complete.□

LEMMA 2.7. *Let $f : X \to Y$ be an approximately quartic mapping satisfying Inequ. 11. Assume in addition a mapping $F : X \to Y$ given by the above-said formula 13. Then $F = F(x)$ is a well-defined mapping and that F is a quartic mapping in X.*

PROOF. Employing Lemma 2.6 and formula 13, one gets that F is a well-defined mapping. This means that the limit 13 exists for all $x \in X$.

In addition claim that F satisfies the functional equation 1 for all 2-dimensional vectors $(x_1, x_2) \in X^2$. In fact, it is clear from the quartic functional inequality 11 and the limit 13 that the following functional inequality

$$\begin{aligned}
&2^{-4n} \| f(2^n x_1 + 2(2^n x_2)) + f(2^n x_1 - 2(2^n x_2)) + 6 f(2^n x_1) \\
&-4 \left[f(2^n x_1 + 2^n x_2) + f(2^n x_1 - 2^n x_2) + 6 f(2^n x_2) \right] \| \leq 2^{-4n} c,
\end{aligned} \tag{28}$$

holds for all vectors $(x_1, x_2) \in X_2$ and all $n \in N$. Therefore from Inequ. 28 one gets

$$\begin{aligned}
&\| \lim_{n \to \infty} 2^{-4n} f\left[2^n (x_1 + 2x_2)\right] + \\
&\lim_{n \to \infty} 2^{-4n} f\left[2^n (x_1 - 2x_2)\right] + 6 \lim_{n \to \infty} 2^{-4n} f(2^n x_1) \\
&-4 \left[\lim_{n \to \infty} 2^{-4n} f\left[2^n (x_1 + x_2)\right] + \right. \\
&\left. \lim_{n \to \infty} 2^{-4n} f\left[2^n (x_1 - x_2)\right] + 6 \lim_{n \to \infty} 2^{-4n} f(2^n x_2) \right] \| \\
&\leq \left(\lim_{n \to \infty} (2^{-4n}) \right) c = 0,
\end{aligned}$$

or

$$\begin{aligned} &\|F(x_1+2x_2)+F(x_1-2x_2)+6F(x_1)- \\ &4\left[F(x_1+x_2)+F(x_1-x_2)+6F(x_2)\right]\| = 0 \end{aligned} \tag{29}$$

or mapping F satisfies the quartic equation 1 for all vectors $(x_1, x_2) \in X^2$. Thus F is a 2-dimensional quartic mapping, completing the proof of Lemma 2.7. □

It is clear now from afore-mentioned Lemmas 1.2-2.7 and especially from general inequality 18, $n \to \infty$, and formula 13 that inequality 12 holds in X. Thus the existence proof in this Theorem is complete.

Proof of uniqueness in Theorem. Let $F' : X \to Y$ be another 2-dimensional quartic mapping satisfying the new quartic functional equation 1, such that inequality

$$\|f(x) - F'(x)\| \leq \frac{17}{180}c, \tag{30}$$

holds for all $x \in X$. If there exists a 2-dimensional quartic mapping $F : X \to Y$ satisfying the new quartic functional equation 1, then

$$F(x) = F'(x), \tag{31}$$

holds for all $x \in X$.

To prove the afore-mentioned uniqueness one employs Equ. 7 for F and F', as well, so that

$$F'(x) = 2^{-4n}F'(2^n x), \tag{32}$$

holds for all $x \in X$, and all $n \in N$. Moreover, the triangle inequality and Ineqs. 12- 30 yield

$$\begin{aligned} &\|F(2^n x) - F'(2^n x)\| \leq \|F(2^n x) - f(2^n x)\| + \|f(2^n x) - F'(2^n x)\|, \\ &\leq \tfrac{17}{180}c + \tfrac{17}{180}c = \tfrac{17}{90}c, \end{aligned}$$

or

$$\|F(2^n x) - F'(2^n x)\| \leq \frac{17}{90}c, \tag{33}$$

for all $x \in X$, and all $n \in N$. Then from Eqs. 7-32, and Inequ. 33, one proves that

$$\begin{aligned} &\|F(x) - F'(x)\| = \|2^{-4n}F(2^n x) - 2^{-4n}F'(2^n x)\| \\ &= 2^{-4n}\|F(2^n x) - F'(2^n x)\| \leq \tfrac{17}{90}2^{-4n}c, \end{aligned} \tag{34}$$

holds for all $x \in X$ and all $n \in N$. Therefore from above Inequ. 34, and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|F(x) - F'(x)\| \leq \frac{17}{90}\left(\lim_{n\to\infty} 2^{-4n}\right)c = 0,$$

or

$$\|F(x) - F'(x)\| = 0,$$

or

$$F(x) = F'(x),$$

for all $x \in X$, completing the proof of uniqueness and thus the stability in this Theorem ([**13**, Theorem 1]) and ([**30**, Theorem 26]).

EXAMPLE. Take $f : R \to R$ be a real function such that $f(x) = x^4 + k$, $k = constant : |k| \leq \frac{c}{24} \left(= \frac{c}{4!}\right)$, in order that f satisfies Inequ. 11. Moreover, there exists a unique quartic real mapping $F : R \to R$ such that from the limit 13 one gets

$$F(x) = \lim_{n\to\infty} 2^{-4n} f(2^n x) = \lim_{n\to\infty} 2^{-4n} \left[(2^n x)^4 + k\right] = x^4.$$

Finally claim that Inequ. 12 holds. In fact, the above condition on k: $|k| \leq \frac{c}{24}$, implies

$$\|f(x) - F(x)\| = \|(x^4 + k) - x^4\| = |k| < \frac{17}{180}c,$$

satisfying Inequ. 12, because from Inequ. 11 one gets that

$$\begin{gathered} \| \left[(x_1 + 2x_2)^4 + k\right] + \left[(x_1 - 2x_2)^4 + k\right] \\ +6\left[x_1^4 + k\right] - 4\left\{\left[(x_1 + x_2)^4 + k\right] + \left[(x_1 - x_2)^4 + k\right] \right. \\ \left. +6\left[x_2^4 + k\right]\right\} \| \leq c, \end{gathered}$$

or

$$|k + k + 6k - 4[k + k + 6k]| = 24|k| \leq c,$$

or

$$|k| \leq \frac{c}{24} = \frac{1}{2}\frac{15}{180}c = \frac{7.5}{180}c < \frac{17}{180}c.$$

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