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SHAPE THEORY OF TRIADS

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ABSTRACT. In this paper we develop the shape theory for triads of spaces in a systematic way, using polyhedral resolutions for triads of spaces, and give applications, which include the Blakers-Massey homotopy excision theorem whose proof is different from the approach taken by Š. Ungar.

1. INTRODUCTION

Throughout the paper, spaces mean topological spaces, and maps mean continuous maps. A *triad of spaces* $(X; X_0, X_1)$ means a space X and two subspaces X_0 and X_1 of X such that $X = X_0 \cup X_1$. A triad of spaces $(X; X_0, X_1)$ is an *ANR triad* if X_0 and X_1 are closed subsets of X and $X, X_0, X_1, X_0 \cap X_1$ are ANR's, and a triad of spaces $(X; X_0, X_1)$ is a *polyhedral triad* (resp., *CW triad*) if X is a polyhedron (resp., CW-complex) and X_0 and X_1 are subpolyhedra (resp., subcomplexes) of X. A *map of triads* $f : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ means a map $f : X \to Y$ such that $f(X_0) \subseteq Y_0$ and $f(X_1) \subseteq Y_1$. A *homotopy of triads* means a map of triads $h : (X \times I; X_0 \times I, X_1 \times I) \to (Y; Y_0, Y_1)$.

In this paper we develop the shape theory for triads of spaces in a systematic way, using polyhedral resolutions for triads of spaces, and give applications. The first application is the Blakers-Massey excision theorem in shape theory. The Blakers-Massey theorem in shape theory was first proved by Ungar [7], but our approach is different and is based on the natural construction of our shape theory of triads. Related results for the excision theorems for strong homology and Čech homology were obtained by Ju. T. Lisica and S. Mardešić [4] and T. Watanabe [8]. As the second application, we obtain the

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Mayer-Vietoris sequences in shape theory for triads of spaces with respect to the Čech cohomology theory based on the normal open coverings.

This paper is organized as follows: After we prove some useful properties of ANR triads in the next section, in section 3 we discuss polyhedral resolutions of triads, and in the following section we obtain results concerning the homotopy types of ANR triads, polyhedral triads and CW triads. In section 5 we show that resolutions can be used to define the shape category for triads, and in the final two sections we discuss invariants in this category and obtain the Blakers-Massey homotopy excision theorem and the Mayer-Vietoris sequences in shape theory.

Let $f, g : X \to Y$ be functions between sets. For any covering $\mathcal{V}$ of Y, $(f, g) < \mathcal{V}$ means that f and g are $\mathcal{V}$-near. For any covering $\mathcal{U}$ of a set X, if A is a subset of X, then $\mathcal{U}|A$ means the covering $\{U \cap A : U \in \mathcal{U}\}$ of A, and the *star* of A in X with respect to $\mathcal{U}$ means the set $\mathrm{st}(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

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2. ANR TRIADS

We will prove some properties of ANR triads that will be needed in later sections. Most of them are analogous to those of single ANR's (see [**6**]).

LEMMA 2.1. *Let $(P; P_0, P_1)$ be an ANR triad. Then, for each open covering $\mathcal{U}$ of P, there exist an open neighborhood W of $P_0 \cap P_1$ in P and a map of triads $k : (P; P_0, P_1) \to (P; P_0, P_1)$ such that $(1_P, k) < \mathcal{U}$ and $k|W$ is a retraction of W onto $P_0 \cap P_1$.*

PROOF. For $i = 0, 1$, [**6**, Lemma 4, p. 86] implies that there exist an open neighborhood V_i of $P_0 \cap P_1$ in P_i and a map $k_i : P_i \to P_i$ so that $(1_{P_i}, k_i) < \mathcal{U}|P_i$ and $k_i|V_i$ is a retraction of V_i onto $P_0 \cap P_1$. Then $V_i = W_i \cap P_i$ for some open subset W_i of P and let $W = W_0 \cap W_1$. Then k_1 and k_2 define a map of triads $k : (P; P_0, P_1) \to (P; P_0, P_1)$ so that $(1_P, k) < \mathcal{U}$ and $k|W$ is a retraction of W onto $P_0 \cap P_1$. □

LEMMA 2.2. *Every ANR triad $(P; P_0, P_1)$ admits an open covering $\mathcal{V}$ of P such that any two $\mathcal{V}$-near maps of triads into $(P; P_0, P_1)$ are homotopic as maps of triads.*

PROOF. [**6**, Theorem 6, p. 39] implies that there exists an open covering $\mathcal{U}$ of P such that any $\mathcal{U}$-near maps $f, g : X \to P$ are homotopic where the homotopy is constant on $x \times I$ whenever $f(x) = g(x)$. By Lemma 2.1, there exist an open neighborhood V of $P_0 \cap P_1$ in P and a map of triads $k : (P; P_0, P_1) \to (P; P_0, P_1)$ such that $(1_P, k) < \mathcal{U}$ and $k|V$ is a retraction of V onto $P_0 \cap P_1$. Now let $\mathcal{U}'$ be the open covering $\{P \setminus P_1, P \setminus P_0, V\}$ of P, and again by [**6**, Theorem 6, p. 39], take an open covering $\mathcal{V}$ of P so that any two $\mathcal{V}$-near maps into P are $\mathcal{U}'$-homotopic. We claim that

$\mathcal{V}$ is a desired open covering. Indeed, let $f, g : (X; X_0, X_1) \to (P; P_0, P_1)$ be $\mathcal{V}$-near maps of triads, and let $G : X \times I \to P$ be the $\mathcal{U}'$-homotopy such that $G_0 = f$ and $G_1 = g$. Then $G(X_0 \times I) \subseteq P \setminus P_1 \cup V$ and $G(X_1 \times I) \subseteq P \setminus P_0 \cup V$, so $H = kG : X \times I \to P$ defines a homotopy of triads $H : (X \times I; X_0 \times I, X_1 \times I) \to (P; P_0, P_1)$ such that $H_0 = kf$ and $H_1 = kg$. On the other hand, by the choice of $\mathcal{U}$, there exist homotopies $K : X_0 \times I \to P_0$ and $K' : X_1 \times I \to P_1$ such that $K_0 = f|X_0$, $K_1 = kf|X_0$, $K'_0 = f|X_1$, $K'_1 = kf|X_1$ and $K|(X_0 \cap X_1) \times t = f|X_0 \cap X_1 = kf|X_0 \cap X_1 = K'|(X_0 \cap X_1) \times t$ for $t \in I$. So the map $\overline{K} : X \times I \to P$ defined by $\overline{K}|X_0 \times I = K$ and $\overline{K}|X_1 \times I = K'$ is a homotopy of triads $\overline{K} : (X \times I; X_0 \times I, X_1 \times I) \to (P; P_0, P_1)$ such that $\overline{K}_0 = f$ and $\overline{K}_1 = kf$, indicating $f \simeq kf$. Similarly, $g \simeq kg$, and hence we have $f \simeq g$ as maps of triads. □

LEMMA 2.3. *Let $(P; P_0, P_1)$ be an ANR triad, let $(X; X_0, X_1)$ be a triad of metric spaces such that X_0, X_1 are closed subsets of X and $X = \mathrm{Int}(X_0) \cup \mathrm{Int}(X_1)$, and let A be a closed subset of X. Then every map of triads $f : (A; A \cap X_0, A \cap X_1) \to (P; P_0, P_1)$ admits an extension $\tilde{f} : (U; U \cap X_0, U \cap X_1) \to (P; P_0, P_1)$ for some open neighborhood U of A in X.*

PROOF. By [**6**, Theorem 10, p. 43], the map of pairs $f|A \cap X_0 : (A \cap X_0, A \cap X_0 \cap X_1) \to (Y_0, Y_0 \cap Y_1)$ extends to a map of pairs $f_0 : (B_0, B_0 \cap X_1) \to (Y_0, Y_0 \cap Y_1)$ for some closed neighborhood B_0 of $A \cap X_0$ in X_0. Consider the map of pairs $f_1 : ((A \cup B_0) \cap X_1, (A \cup B_0) \cap X_1 \cap X_0) \to (Y_1, Y_0 \cap Y_1)$ defined by $f_1|A \cap X_1 = f|A \cap X_1$ and $f_1|B_0 \cap X_1 = f_0|B_0 \cap X_1$. Again by [**6**, Theorem 10, p. 43], f_1 extends to a map of pairs $f'_1 : (B_1, B_1 \cap X_0) \to (Y_1, Y_0 \cap Y_1)$ for some closed neighborhood B_1 of $(A \cup B_0) \cap X_1$ in X_1. Now let $U' = B_0 \cup B_1$, and define a map of triads $\tilde{f}' : (U'; U' \cap X_0, U' \cap X_1) \to (Y; Y_0, Y_1)$ by $\tilde{f}'|B_0 = f_0$ and $\tilde{f}'|B_1 = f'_1$. Then since $X = \mathrm{Int}(X_0) \cup \mathrm{Int}(X_1)$, U' is a closed neighborhood of A in X. Finally, if U is an open subset of X such that $A \subseteq U \subseteq U'$, then $\tilde{f} = \tilde{f}'|U$ is a desired map of triads. □

LEMMA 2.4. *Let $(P; P_0, P_1), (X; X_0, X_1)$ and A be as in Lemma 2.3, and let $f, g : (X; X_0, X_1) \to (P; P_0, P_1)$ be maps of triads. If $f|A \simeq g|A$ as maps of triads from $(A; A \cap X_0, A \cap X_1)$ to $(P; P_0, P_1)$, then there exists an open neighborhood V of A in X such that $f|V \simeq g|V$ as maps of triads from $(V; V \cap X_0, X_1)$ to $(P; P_0, P_1)$.*

PROOF. Let $H : (A \times I; (A \cap X_0) \times I, (A \cap X_1) \times I) \to (P; P_0, P_1)$ be a homotopy of triads such that $H_0 = f|A$ and $H_1 = g|A$, let $B = (A \times I) \cup (X \times 0) \cup (X \times 1)$, and define a map of triads $F : (B; B \cap (X_0 \times I), B \cap (X_1 \times I)) \to (P; P_0, P_1)$ by $F|A \times I = H, F|X \times 0 = f$ and $F|X \times 1 = g$. Applying Lemma 2.3, F extends to $\tilde{F} : (U; U \cap (X_0 \times I), U \cap (X_1 \times I)) \to (Y; Y_0, Y_1)$ for some open neighborhood U of B. If V is an open set such that $V \times I \subseteq U$, then $\tilde{H} = \tilde{F}|V \times I$ is a desired homotopy. □

LEMMA 2.5. *Let $(X; X_0, X_1)$ be a triad of spaces, let $(P; P_0, P_1)$ and $(P'; P_0', P_1')$ be ANR triads, and let $f : (X; X_0, X_1) \to (P'; P_0', P_1')$ and $g_1, g_2 : (P'; P_0', P_1') \to (P; P_0, P_1)$ be maps of triads such that $g_1 f \simeq g_2 f$ as maps of triads. Then there exist an ANR triad $(P''; P_0'', P_1'')$ and maps of triads $f' : (X; X_0, X_1) \to (P''; P_0'', P_1'')$ and $g : (P''; P_0'', P_1'') \to (P'; P_0', P_1')$ such that $f = gf'$ and $g_1 g \simeq g_2 g$ as maps of triads.*

PROOF. We can prove this by the argument similar to [**6**, Lemma 2, p. 52], using Lemma 2.4 in an appropriate place. □

LEMMA 2.6. *Let $(X; X_0, X_1)$ be a triad of spaces where X is normal, let A be a closed subset of X, and let V be an open neighborhood of A in X. Then there exists a map of triads*

$$r : (X \times I; X_0 \times I, X_1 \times I) \to (V \times I \cup X \times 0; (V \cap X_0) \times I \cup X_0 \times 0, (V \cap X_1) \times I \cup X_1 \times 0)$$

such that the restriction $r|A \times I \cup X \times 0$ is the inclusion.

LEMMA 2.7. *(Homotopy extension lemma) Let $(X; X_0, X_1)$ and $A \subseteq X$ be as in Lemma 2.6, and let $(Y; Y_0, Y_1)$ be an ANR triad. If $f, g : (A; A \cap X_0, A \cap X_1) \to (Y; Y_0, Y_1)$ are homotopic maps of triads, and if g extends to a map of triads $\tilde{g} : (X; X_0, X_1) \to (Y; Y_0, Y_1)$, then there is an extension $\tilde{f} : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ of f such that $\tilde{f} \simeq \tilde{g}$ as maps of triads.*

PROOF. We can proceed as for [**6**, Theorem 9, p. 41], using Lemma 2.6. □

3. RESOLUTIONS OF TRIADS

Let **Top** be the category of spaces and maps, and let $\mathbf{Top}^T$ be the category of triads of spaces and maps of triads. Recall that a *resolution* of a triad $(X; X_0, X_1)$ is a morphism $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ in $\textbf{pro-Top}^T$ with the following two properties [**5**]:

- **(R1):** Let $(P; P_0, P_1)$ be an ANR triad, and let $\mathcal{V}$ be an open covering of P. Then every map of triads $f : (X; X_0, X_1) \to (P; P_0, P_1)$ admits $\lambda \in \Lambda$ and a map of triads $g : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P; P_0, P_1)$ such that $(gp_\lambda, f) < \mathcal{V}$; and
- **(R2):** Let $(P; P_0, P_1)$ be an ANR triad. Then for each open covering $\mathcal{V}$ of P there exists an open covering $\mathcal{V}'$ of P such that whenever $\lambda \in \Lambda$ and $g, g' : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P; P_0, P_1)$ are maps of triads such that $(gp_\lambda, g'p_\lambda) < \mathcal{V}'$, then $(gp_{\lambda\lambda'}, g'p_{\lambda\lambda'}) < \mathcal{V}$ for some $\lambda' \geq \lambda$.

$\boldsymbol{p}$ is an *ANR-resolution* (resp., *polyhedral resolution*) if $(X_\lambda; X_{0\lambda}, X_{1\lambda})$ are all ANR triads (resp., polyhedral triads). The pointed version of resolution is also defined similarly.

THEOREM 3.1. *(Mardešić [5]) Every triad $(X; X_0, X_1)$ of spaces admits an ANR-resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that Λ is cofinite and $X_\lambda = \mathrm{Int}(X_{0\lambda}) \cup \mathrm{Int}(X_{1\lambda})$ for each $\lambda \in \Lambda$.

In this section, we wish to show the following theorem, which we will need in later sections.

THEOREM 3.2. *Every triad $(X; X_0, X_1)$ of spaces admits a polyhedral resolution $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ such that Λ is cofinite.*

To prove the theorem, we need a couple of lemmas.

LEMMA 3.3. *Let $(X; X_0, X_1)$ be a triad of spaces, and let*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a morphism in **pro-Top**^{T} *such that the induced morphism $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$ is a resolution, and the induced morphisms $\boldsymbol{p}|X_0 = (p_\lambda|X_0) : X_0 \to \boldsymbol{X}_0$ and $\boldsymbol{p}|X_1 = (p_\lambda|X_1) : X_1 \to \boldsymbol{X}_1$ in* **pro-Top** *satisfy property (B1):*

(B1): *Let $\lambda \in \Lambda$, and let U be an open subset of X_λ such that $\mathrm{Cl}(p_\lambda(X)) \subseteq U$. Then there exists $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq U$.*

Then $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ is a resolution.

PROOF. Clearly, (R2) for $\boldsymbol{p} : X \to \boldsymbol{X}$ implies (R2) for $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$. So it suffices to verify (R1). Let $(P; P_0, P_1)$ be an ANR triad, let $h : (X; X_0, X_1) \to (P; P_0, P_1)$ be a map of triads, and let $\mathcal{V}$ be an open covering of P. Let $\mathcal{V}'$ be an open covering of P such that $\mathrm{st}\,\mathcal{V}' < \mathcal{V}$. Apply Lemma 2.1 to $\mathcal{V}'$, we obtain an open neighborhood W of $P_0 \cap P_1$ in P and a map of triads $k : (P; P_0, P_1) \to (P; P_0, P_1)$ such that $k|W : W \to P_0 \cap P_1$ is a retraction and $(1_P, k) < \mathcal{V}'$. Take an open set W' such that $P_0 \cap P_1 \subseteq W' \subseteq \mathrm{Cl}(W') \subseteq W$, and let $\mathcal{V}''$ be an open covering of P such that $\mathcal{V}'' < \mathcal{V}' \wedge \{W', P \setminus P_0, P \setminus P_1\}$. By (R1) for $\boldsymbol{p} : X \to \boldsymbol{X}$, there exist $\lambda \in \Lambda$ and a map $f : X_\lambda \to P$ such that $(h, fp_\lambda) < \mathcal{V}''$. Then $fp_\lambda(X_0) \subseteq W' \cup P \setminus P_1$ and $fp_\lambda(X_1) \subseteq W' \cup P \setminus P_0$. So, $f(\mathrm{Cl}(p_\lambda(X_0))) \subseteq \mathrm{Cl}(W') \cup \mathrm{Cl}(P \setminus P_1) \subseteq W \cup P_0$, and so $\mathrm{Cl}(p_\lambda(X_0)) \subseteq f^{-1}(W \cup P_0)$. Similarly, $\mathrm{Cl}(p_\lambda(X_1)) \subseteq f^{-1}(W \cup P_1)$. Since $W \cup P_0$ and $W \cup P_1$ are open, (B1) for $\boldsymbol{p}|X_0 : X_0 \to \boldsymbol{X}_0$ and $\boldsymbol{p}|X_1 : X_1 \to \boldsymbol{X}_1$ imply that there exists $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{0\lambda'}) \subseteq f^{-1}(W \cup P_0)$ and $p_{\lambda\lambda'}(X_{1\lambda'}) \subseteq f^{-1}(W \cup P_1)$. Now let $f' : X_{\lambda'} \to P$ be defined by $f' = kfp_{\lambda\lambda'}$. Then $f'(X_{0\lambda'}) = kfp_{\lambda\lambda'}(X_{0\lambda'}) \subseteq P_0$, and similarly $f'(X_{1\lambda'}) \subseteq P_1$. So f' defines a map of triads $f' : (X_{\lambda'}; X_{0\lambda'}, X_{1\lambda'}) \to (P; P_0, P_1)$ satisfies $(f'p_\lambda, h) < \mathcal{V}$. This verifies (R2) for $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$. □

LEMMA 3.4. *Let $(X; X_0, X_1)$ be a triad of spaces, and let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a morphism in* **pro-Top**. *For each $\lambda \in$*

Λ, let M_λ be the index set for all open coverings $\mathcal{V}_{\lambda,\mu}$ of X_λ, and let $M = \{\nu = (\lambda,\mu) : \lambda \in \Lambda, \mu \in M_\lambda\}$. For each $\nu = (\lambda,\mu) \in M$, let $(Z_\nu; Z_{0\nu}, Z_{1\nu}) = (X_\lambda; \mathrm{st}(p_\lambda(X_0), \mathcal{V}_{\lambda,\mu}), \mathrm{st}(p_\lambda(X_1), \mathcal{V}_{\lambda,\mu}))$, and order M by $\nu = (\lambda,\mu) \le \nu' = (\lambda',\mu')$ provided $\lambda \le \lambda'$ and $p_{\lambda\lambda'}(Z_{i\nu}) \subseteq Z_{i\nu'}$, $i = 0,1$. Now let $r_\nu = p_\lambda : (X; X_0, X_1) \to (Z_\nu; Z_{0\nu}, Z_{1\nu})$ for each $\nu \in M$, and let $r_{\nu\nu'} = p_{\lambda\lambda'} : (Z_{\nu'}; Z_{0\nu'}, Z_{1\nu'}) \to (Z_\nu; Z_{0\nu}, Z_{1\nu})$ for $\nu \le \nu'$. Then if $\boldsymbol{p}$ is a resolution, then so is the morphism

$$\boldsymbol{r} = (r_\nu) : (X; X_0, X_1) \to (\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1) = ((Z_\nu; Z_{0\nu}, Z_{1\nu}), r_{\nu\nu'}, M)$$

in **pro-Top**^{T}.

PROOF. It is easy to see that $\boldsymbol{r}|X_0 = (r_\nu|X_0) : X_0 \to \boldsymbol{Z}_0$ and $\boldsymbol{r}|X_1 = (r_\nu|X_1) : X_1 \to \boldsymbol{Z}_1$ satisfy property (B1). If $\boldsymbol{p}$ is a resolution, then so is $\boldsymbol{r}|X = (r_\nu) : X \to \boldsymbol{Z}$. Lemma 3.3 implies that $\boldsymbol{r} : (X; X_0, X_1) \to (\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1)$ is a resolution. □

LEMMA 3.5. *Let X be a polyhedron, and let A and B be closed subsets of X such that $X = A \cup B$. Then for any open sets U_0 and U_1 in X with $A \subseteq U_0$ and $B \subseteq U_1$, there exists a polyhedral triad $(X; X_0, X_1)$ such that $A \subseteq \mathrm{Int}(X_0) \subseteq X_0 \subseteq U_0$ and $B \subseteq \mathrm{Int}(X_1) \subseteq X_1 \subseteq U_1$.*

PROOF OF THEOREM 3.2. There exists a polyhedral resolution $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ with cofinite index set Λ (see [**6**, Theorem 7, p. 84]). For this $\boldsymbol{p}$, we have a resolution $\boldsymbol{r} = (r_\nu) : (X; X_0, X_1) \to (\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1) = ((Z_\nu; Z_{0\nu}, Z_{1\nu}), r_{\nu\nu'}, M)$ as in Lemma 3.4. Let N be the subset of M so that each $\nu \in N$ corresponds to a polyhedral triad $(Z_\nu; Z_{0\nu}, Z_{1\nu})$ as in Lemma 3.5. Here note that we can assume that each M_λ in Lemma 3.5 is cofinite, and hence N is cofinite. Then the induced morphism $\boldsymbol{r} = (r_\nu) : (X; X_0, X_1) \to (\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1) = ((Z_\nu; Z_{0\nu}, Z_{1\nu}), r_{\nu\nu'}, N)$ is a desired resolution. □

We also have the pointed analog of Theorem 3.2.

THEOREM 3.6. *Every triad $(X; X_0, X_1, x_0)$ of spaces with a base point admits a polyhedral resolution $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1, x_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, x_0) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$ with a cofinite index set Λ.*

PROOF. The pointed versions of Lemmas 3.3 and 3.4 hold. Thus the theorem follows from the following lemma. □

LEMMA 3.7. *Let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a resolution, and let $x_0 \in X$. Then the morphism*

$$\boldsymbol{p} = (p_\lambda) : (X, x_0) \to (\boldsymbol{X}, x_0) = ((X_\lambda, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

where $x_{0\lambda} = p_\lambda(x_0)$ is a resolution.

PROOF. (R2) for $\boldsymbol{p} : X \to \boldsymbol{X}$ implies (R2) for $\boldsymbol{p} : (X, x_0) \to (\boldsymbol{X}, x_0)$, so it suffices to verify (R1). Let (P, p_0) be a pointed ANR, and let $g : (X, x_0) \to$

(P, p_0) be a pointed map. Let $\mathcal{V}$ be any open covering of P, and take an open covering $\mathcal{V}'$ of P such that $\operatorname{st}\mathcal{V}' < \mathcal{V}$. [**6**, Lemma 4, p. 86] implies that there exist an open neighborhood W of p_0 in P and a map $k : P \to P$ such that $(1_P, k) < \mathcal{V}'$ and $k|W : W \to \{p_0\}$ is a retraction. Now let $\mathcal{V}''$ be an open covering of P such that $\mathcal{V}'' < \mathcal{V}' \wedge \{W, P \setminus \operatorname{Cl}(W')\}$ where W' is an open set such that $p_0 \in W' \subseteq \operatorname{Cl}(W') \subseteq W$. By (R1) for $\boldsymbol{p} : X \to \boldsymbol{X}$, there exist $\lambda \in \Lambda$ and a map $h : X_\lambda \to P$ such that $(g, hp_\lambda) < \mathcal{V}''$. Let $h' = kh : X_\lambda \to P$. Then h' defines a pointed map $h' : (X_\lambda, x_{0\lambda}) \to (P, p_0)$ and $(g, h'p_\lambda) < \operatorname{st}\mathcal{V}' < \mathcal{V}$. This verifies (R1) for $\boldsymbol{p} : (X, x_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$. □

LEMMA 3.8. *Let $(X; X_0, X_1)$ be a triad of spaces such that X_0 and X_1 are closed subsets of X, and let $\mathcal{U}$ be a covering of X by path-connected subsets of X. Then* $\operatorname{st}(X_0, \mathcal{U}) \cap \operatorname{st}(X_1, \mathcal{U}) = \operatorname{st}(X_0 \cap X_1, \mathcal{U})$.

PROOF. Let $x \in \operatorname{st}(X_0, \mathcal{U}) \cap \operatorname{st}(X_1, \mathcal{U})$. Without loss of generality, let $x \in X_0$. Since $x \in \operatorname{st}(X_1, \mathcal{U})$, there is $U \in \mathcal{U}$ such that $x \in U$ and $U \cap X_1 \neq \emptyset$. Then if $U \cap X_0 \cap X_1 = \emptyset$, this would contradict the connectedness of the unit interval I. Indeed, let $x' \in U \cap X_1$. Then for any path $\varphi : I \to U$ with $\varphi(0) = x$ and $\varphi(1) = x'$, I would be the disjoint union of the nonempty closed subsets $\varphi^{-1}(U \cap X_0)$ and $\varphi^{-1}(U \cap X_1)$. So $U \cap X_0 \cap X_1 \neq \emptyset$. Thus $x \in \operatorname{st}(X_0 \cap X_1, \mathcal{U})$. The other inclusion is obvious. □

THEOREM 3.9. *Every triad $(X; X_0, X_1)$ of spaces such that X_0 and X_1 are normally embedded closed subsets of X admits a polyhedral resolution $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ with Λ being cofinite such that the induced morphisms $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$, $\boldsymbol{p}|X_i = (p_\lambda|X_i) : X_i \to \boldsymbol{X}_i$, $i = 0, 1$, and $\boldsymbol{p}|X_0 \cap X_1 = (p_\lambda|X_0 \cap X_1) : X_0 \cap X_1 \to \boldsymbol{X}_0 \cap \boldsymbol{X}_1 = (X_{0\lambda} \cap X_{1\lambda}, p_{\lambda\lambda'}|X_{0\lambda'} \cap X_{1\lambda'}, \Lambda)$ are resolutions.*

PROOF. Indeed, let

$$\boldsymbol{r} : (X; X_0, X_1) \to (\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1) = ((Z_\nu; Z_{0\nu}, Z_{1\nu}), r_{\nu\nu'}, M)$$

be the polyhedral resolution obtained as in the proof of Theorem 3.2. Then the restrictions $\boldsymbol{r}|X_i : X_i \to \boldsymbol{Z}_i$, $i = 0, 1$, are resolutions as in [**6**, Theorem 11, p. 89]. Note that for each $\nu = (\lambda, \mu) \in M$ and $i = 0, 1$, $Z_{i\nu} = \operatorname{st}(p_\lambda(X_i), \mathcal{V}_{\lambda,\mu})$ for some open covering $\mathcal{V}_{\lambda,\mu}$ that is a star covering with respect to some subdivision of $X_{i\lambda}$. Then by Lemma 3.8 the induced morphism $\boldsymbol{r}|X_0 \cap X_1 = (r_\nu|X_0 \cap X_1) : X_0 \cap X_1 \to \boldsymbol{Z}_0 \cap \boldsymbol{Z}_1 = (Z_{0\nu} \cap Z_{1\nu}, r_{\nu\nu'}|Z_{0\nu} \cap Z_{1\nu}, M)$ forms a resolution as in [**6**, Theorem 11, p. 89]. □

4. THE HOMOTOPY TYPES OF ANR TRIADS

We first show

THEOREM 4.1. *Every ANR triad is homotopy dominated by some polyhedral triad.*

PROOF. Let $(X;X_0,X_1)$ be an ANR triad. Take an open covering $\mathcal{V}$ of X so that any two $\mathcal{V}$-near maps of triads to $(X;X_0,X_1)$ are homotopic (Lemma 2.2), and also take a polyhedral resolution $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)=((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$ (Theorem 3.2). Then there exist $\lambda\in\Lambda$ and a map of triads $g:(X_\lambda;X_{0\lambda},X_{1\lambda})\to(X;X_0,X_1)$ such that $(1_X,gp_\lambda)<\mathcal{V}$, and hence $1_X\simeq gp_\lambda$ as maps of triads. □

The following is an analog of J. H. C. Whitehead's classical theorem [**9**]:

THEOREM 4.2. *Let $(X;X_0,X_1)$ be a triad of spaces such that $X=\mathrm{Int}(X_0)\cup\mathrm{Int}(X_1)$. If $(X;X_0,X_1)$ is homotopy dominated by a polyhedral triad, then $(X;X_0,X_1)$ has the homotopy type of a polyhedral triad.*

We can prove the theorem analogously to the proof of [**6**, Theorem 3, p. 315], using the two lemmas in the below.

We call the map of triads $\varphi:(X;X_0,X_1)\to(Y;Y_0,Y_1)$ a *weak homotopy equivalence* if $\varphi:X\to Y$, $\varphi|X_0:X_0\to Y_0$, $\varphi|X_1:X_1\to Y_1$ and $\varphi|X_0\cap X_1:X_0\cap X_1\to Y_0\cap Y_1$ are all weak homotopy equivalences.

LEMMA 4.3. *Let $(X;X_0,X_1)$ be a triad of spaces such that $X=\mathrm{Int}(X_0)\cup\mathrm{Int}(X_1)$. Then there exist a polyhedral triad $(P;P_0,P_1)$ and a weak homotopy equivalence $\varphi:(P;P_0,P_1)\to(X;X_0,X_1)$.*

PROOF. As in [**6**, Theorem 10, p. 321], we have polyhedral pairs (P_0,P_{01}) and (P_1,P_{01}) and maps of triads $\varphi_0:(P_0,P_{01})\to(X_0,X_0\cap X_1)$ and $\varphi_1:(P_1,P_{01})\to(X_1,X_0\cap X_1)$ such that $\varphi_0|P_{01}=\varphi_1|P_{01}$ and $\varphi_0,\varphi_1,\varphi_0|P_{01}$ are all weak homotopy equivalences. Let $P=P_0\cup P_1$, and let $\varphi:(P;P_0,P_1)\to(X;X_0,X_1)$ be the map of triads such that $\varphi|P_0=\varphi_0$ and $\varphi|P_1=\varphi_1$. Then by [**2**, 16.24], $\varphi:P\to X$ is a weak homotopy equivalence. □

LEMMA 4.4. *Let $\varphi:(X;X_0,X_1)\to(Y;Y_0,Y_1)$ be a weak homotopy equivalence. Then for each polyhedral triad $(P;P_0,P_1)$, the induced map $\varphi_*:[(P;P_0,P_1),(X;X_0,X_1)]\to[(P;P_0,P_1),(X;X_0,X_1)]$ is a bijection. Here $[\ ,\ ]$ denotes the set of homotopy classes.*

PROOF. We can easily modify the proof of [**2**, 16.20], so the proof is omitted. □

The following is an immediate consequence of Theorems 4.1 and 4.2.

THEOREM 4.5. *Let $(X;X_0,X_1)$ be a triad of spaces such that $X=\mathrm{Int}(X_0)\cup\mathrm{Int}(X_1)$. Then the following statements are equivalent:*

i) *$(X;X_0,X_1)$ has the homotopy type of a polyhedral triad;*
ii) *$(X;X_0,X_1)$ has the homotopy type of a CW triad;*
iii) *$(X;X_0,X_1)$ has the homotopy type of an ANR triad;*
iv) *$(X;X_0,X_1)$ is homotopy dominated by a polyhedral triad;*
v) *$(X;X_0,X_1)$ is homotopy dominated by a CW triad;*

vi) $(X; X_0, X_1)$ *is homotopy dominated by an ANR triad.*

REMARK. The pointed versions of Theorems 4.1, 4.2 and 4.5 also hold.

5. SHAPE OF TRIADS

Let $\mathbf{HTop}^T$ be the category of triads of spaces and homotopy classes of maps of triads, and let $\mathbf{HPol}^T$ be the full subcategory of $\mathbf{HTop}^T$ whose objects are the triads of spaces which have the homotopy type of a polyhedral triad. The corresponding pointed categories are denoted by $\mathbf{HTop}_*^T$ and $\mathbf{HPol}_*^T$.

THEOREM 5.1. *Every polyhedral resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

induces an $\mathbf{HPol}^T$*-expansion*

$$\begin{aligned} H\boldsymbol{p} = (Hp_\lambda) &: (X; X_0, X_1) \to H(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \\ &= ((HX_\lambda; HX_{0\lambda}, HX_{1\lambda}), Hp_{\lambda\lambda'}, \Lambda) \end{aligned}$$

Here H denotes the functor from the topological category to the homotopy category.

PROOF. We must verify properties (E1) and (E2) of [**6**]. Property (E1) follows from property (R1) if we take an open covering $\mathcal{V}$ as in Lemma 2.2. For property (E2), we proceed as for [**6**, Theorem 2, p. 75], taking $\mathcal{V}$ as in Lemma 2.2 and using Lemma 2.5 in the place of [**6**, Lemma 1, p. 46]. □

By Theorems 3.2 and 5.1, the pair of categories $(\mathbf{HTop}^T, \mathbf{HPol}^T)$ defines a shape category, which we call the *shape category of triads* and denote by $\mathbf{Sh}^T$. Lemmas 2.2 and 2.5 hold in the pointed case, and so the pointed analog of Theorem 5.1 holds. This and Theorem 3.6 imply that the pair of categories $(\mathbf{HTop}_*^T, \mathbf{HPol}_*^T)$ defines a shape category, which we call the *pointed shape category of triads* and denote by $\mathbf{Sh}_*^T$.

6. EXCISION THEOREM IN SHAPE THEORY

Throughout this section, all triads are assumed to have base points, and we do not write the base points. For each triad of spaces $(X; X_0, X_1)$ and for $k \geq 2$, we define the *k-th homotopy pro-set of triad* pro-$\pi_k(X; X_0, X_1)$ as the pro-set $\pi_k(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = (\pi_k(X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'*}, \Lambda)$ where $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ is an $\mathbf{HPol}_*^T$- expansion. For each morphism $\varphi : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ in $\mathbf{Sh}_*^T$, there is an induced morphism pro-$\pi_k(\varphi)$: pro-$\pi_k(X; X_0, X_1) \to$ pro-$\pi_k(Y; Y_0, Y_1)$. Then pro-π_k defines a functor from $\mathbf{Sh}_*^T$ to **pro-Ab** for $k \geq 4$, to **pro-Gp** for $k = 3$, and to **pro-Set** for $k = 2$, where **Gp** is the category of groups and homomorphisms, **Ab** is the full subcategory of **Gp** whose objects are abelian groups, and **Set** is the category of pointed sets and point preserving functions.

THEOREM 6.1. *Let $(X;X_0,X_1)$ be a triad of spaces such that X is normal and X_0 and X_1 are normally embedded closed subspaces of X. Then there exist exact sequences of pro-sets*

$$\begin{gathered}\to \text{pro-}\pi_{r+1}(X;X_0,X_1) \xrightarrow{\partial} \text{pro-}\pi_r(X_0,X_0\cap X_1) \xrightarrow{i_*} \\ \text{pro-}\pi_r(X,X_1) \xrightarrow{j_*} \text{pro-}\pi_r(X;X_0,X_1) \to\end{gathered}$$

$$\cdots \to \text{pro-}\pi_2(X;X_0,X_1) \xrightarrow{\partial} \text{pro-}\pi_1(X_0,X_0\cap X_1) \xrightarrow{i_*} \text{pro-}\pi_1(X,X_1)$$

and

$$\begin{gathered}\to \text{pro-}\pi_{r+1}(X;X_0,X_1) \xrightarrow{\partial'} \text{pro-}\pi_r(X_1,X_0\cap X_1) \xrightarrow{i'_*} \\ \text{pro-}\pi_r(X,X_0) \xrightarrow{j'_*} \text{pro-}\pi_r(X;X_0,X_1) \to\end{gathered}$$

$$\cdots \to \text{pro-}\pi_2(X;X_0,X_1) \xrightarrow{\partial'} \text{pro-}\pi_1(X_1,X_0\cap X_1) \xrightarrow{i'_*} \text{pro-}\pi_1(X,X_0)$$

PROOF. Let

$$\boldsymbol{p} = (p_\lambda) : (X;X_0,X_1) \to (\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1) = ((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$$

be a polyhedral resolution (Theorem 3.6). Then [**5**, Section 5] implies that the induced morphisms

$$\begin{cases} \boldsymbol{p} = (p_\lambda) : (X,X_i) \to (\boldsymbol{X},\boldsymbol{X}_i) = ((X_\lambda,X_{i\lambda}),p_{\lambda\lambda'},\Lambda) \\ \boldsymbol{p}|X_i = (p_\lambda|X_i) : (X_i,X_0\cap X_1) \to (\boldsymbol{X}_i,\boldsymbol{X}_0\cap \boldsymbol{X}_1) = \\ ((X_{i\lambda},X_{0\lambda}\cap X_{1\lambda}),p_{\lambda\lambda'}|X_{i\lambda},\Lambda) \end{cases}$$

are resolutions for $i=0,1$, and hence [**6**, Theorem 8, p. 86] implies that those resolutions induce expansions

$$\begin{cases} H\boldsymbol{p} = (Hp_\lambda) : (X,X_i) \to H(\boldsymbol{X},\boldsymbol{X}_i) = ((X_\lambda,X_{i\lambda}),Hp_{\lambda\lambda'},\Lambda) \\ H\boldsymbol{p}|X_i = (Hp_\lambda|X_i) : (X_i,X_0\cap X_1) \to H(\boldsymbol{X}_i,\boldsymbol{X}_0\cap \boldsymbol{X}_1) = \\ ((X_{i\lambda},X_{0\lambda}\cap X_{1\lambda}),Hp_{\lambda\lambda'}|X_{i\lambda},\Lambda) \end{cases}$$

for $i=0,1$. So the homotopy sequences of the triad $(X_\lambda;X_{0\lambda},X_{1\lambda})$ (see [**3**, p.160]) and their naturality give rise to the above exact sequences of pro-sets by the pro-set version of [**6**, Theorem 10, p. 119]. □

THEOREM 6.2. *Let $(X;X_0,X_1)$ be a triad of spaces such that X is normal and X_0 and X_1 are normally embedded closed subspaces of X, and let $m\geq 2$. Then the inclusion induced morphism*

$$i_* : \text{pro-}\pi_r(X_0,X_0\cap X_1) \to \text{pro-}\pi_r(X,X_1)$$

is an isomorphism for $2\leq r<m$, an epimorphism for $r=m$ and "monic" for $r=1$ i.e., $\mathrm{Ker}\{i_* : \text{pro-}\pi_1(X_0,X_0\cap X_1) \to \text{pro-}\pi_1(X,X_1)\}\approx 0$, *if and only if* $\text{pro-}\pi_r(X;X_0,X_1)\approx 0$ *for $2\leq r\leq m$.*

PROOF. This is an immediate consequence of Theorem 6.1. □

THEOREM 6.3. *(Blakers-Massey theorem in shape theory) Let* $(X; X_0, X_1)$ *be a triad of spaces such that* X *is normal and* X_0 *and* X_1 *are normally embedded connected closed subspaces of* X, *and let* $m, n \geq 1$. *Then if* $(X_0, X_0 \cap X_1)$ *is* n*-shape connected and* $(X_1, X_0 \cap X_1)$ *is* m*-shape connected, then the inclusion induced morphism*

$$i_* : \text{pro-}\pi_r(X_0, X_0 \cap X_1) \to \text{pro-}\pi_r(X, X_1)$$

is an isomorphism for $1 \leq r \leq n+m-1$ *and an epimorphism for* $r = n+m$.

We prove the following two lemmas before we prove the theorem.

LEMMA 6.4. *Let* $1 \leq n \leq m$, *let* $(X_i; A_i, B_i)$, $i = 0, 1, \ldots, m$, *be polyhedral triads such that* A_0 *and* B_0 *are connected, and let* $p_i : (X_i; A_i, B_i) \to (X_{i+1}; A_{i+1}, B_{i+1})$, $i = 0, 1, \ldots, m$, *be maps of triads such that the induced maps* $(p_i|A_i)_* : \pi_i(A_i, A_i \cap B_i) \to \pi_i(A_{i+1}, A_{i+1} \cap B_{i+1})$ *for* $i = 0, 1, \ldots, n$ *and* $(p_i|B_i)_* : \pi_i(B_i, A_i \cap B_i) \to \pi_i(B_{i+1}, A_{i+1} \cap B_{i+1})$ *for* $i = 0, 1, \ldots, m$ *are trivial. Then there exist a polyhedral triad* $(P; P', P'')$ *such that* $(P', P' \cap P'')$ *is* n*-connected and* $(P'', P' \cap P'')$ *is* m*-connected, and maps of triads* $f : (X_0; A_0, B_0) \to (P; P', P'')$ *and* $g : (P; P', P'') \to (X_m; A_m, B_m)$ *such that* $p_n \cdots p_1 p_0 = gf$.

PROOF. Let $\left\{ \begin{array}{c} (K_1, L) \\ (K_2, L) \end{array} \right\}$ be triangulations of $\left\{ \begin{array}{c} (A_0, A_0 \cap B_0) \\ (B_0, A_0 \cap B_0) \end{array} \right\}$ such that L is a full subcomplex of K_1 and also of K_2. For each $i = 0, 1, \ldots, m$, let

$$\left\{ \begin{array}{l} Q_i = ((A_0 \cap B_0) \times I) \cup ((|K_1^{\min\{i,n\}}| \cup |K_2^i|) \times I) \\ P_i = Q_i \cup (A_0 \times 0) \\ P_i' = Q_i \cup (B_0 \times 0) \end{array} \right.$$

Then the polyhedral pairs $\left\{ \begin{array}{c} (P_1, Q_i) \\ (P_i', Q_i) \end{array} \right\}$ respectively have the homotopy types of the polyhedral pairs

$$\left\{ \begin{array}{c} (|K_1| \cup |K_2^i|, |L| \cup |K_1^{\min\{i,n\}}| \cup |K_2^i|) \\ (|K_1^{\min\{i,n\}}| \cup |K_2|, |L| \cup |K_1^{\min\{i,n\}}| \cup |K_2^i|) \end{array} \right\}$$

So for $i = 0, 1, \ldots, m$, (P_i, Q_i) is $\min\{i, n\}$-connected, and (P_i', Q_i) is i-connected.

We wish to obtain the following commutative diagram:

$$\begin{tikzcd}
(X_0; A_0, B_0) \arrow[r, "p_0"] \arrow[d, equal] & (X_1; A_1, B_1) \arrow[r, "p_1"] & \cdots \\
(X_0; A_0, B_0) \arrow[r, "\subseteq"] & (P_0 \cup P_0'; P_0, P_0') \arrow[u, "g_0"] \arrow[r, "\subseteq"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "p_n"] & (X_{n+1}; A_{n+1}, B_{n+1}) \arrow[r, "p_{n+1}"] & \cdots \arrow[r, "p_m"] & (X_{m+1}; A_{m+1}, B_{m+1}) \\
\cdots \arrow[r, "\subseteq"] & (P_n \cup P'_n; P_n, P'_n) \arrow[u, "g_n"] \arrow[r, "\subseteq"] & \cdots \arrow[r, "\subseteq"] & (P_m \cup P'_m; P_m, P'_m) \arrow[u, "g_{m-1}"']
\end{tikzcd}$$

We can proceed as in [**6**, Lemma 3, p. 140]. For g_0, let $g_0|X_0 \times 0 = p_0$ and $g_0(x \times I) = p_0(x)$ for $x \in A_0 \cap B_0$, and using the hypothesis that A_0 and B_0 are connected, for each vertex v of $K_1 \setminus L \cup K_2 \setminus L$, let $g_0(v \times I)$ be a path in A_1 or B_1 from $g_0(v,0) = p_0(v)$ to $g_0(v,1) =$ the base point of $(X_1; A_1, B_1)$. Assume we have defined g_{i-1} for some $i \leq m$. Then for each i-simplex σ of

$$\left\{ \begin{array}{ll} K_1 \setminus L \cup K_2 \setminus L & \text{for } i \leq n \\ K_2 \setminus L & \text{for } n < i \leq m \end{array} \right\}$$

the pair $((\partial\sigma \times I) \cup (\sigma \times 0), \partial\sigma \times 1)$ is an i-cell in

$$\left\{ \begin{array}{ll} P_{i-1} \text{ or } P'_{i-1} & \text{for } i \leq n \\ P'_{i-1} & \text{for } n < i \leq m \end{array} \right\}$$

with its boundary in Q_{i-1}. Then use the hypothesis that $(p_i|A_i)_* = 0 : \pi_i(A_i, A_i \cap B_i) \to \pi_i(A_{i+1}, A_{i+1} \cap B_{i+1})$ $(i = 0, 1, \ldots, n)$ and $(p_i|B_i)_* = 0 : \pi_i(B_i, A_i \cap B_i) \to \pi_i(B_{i+1}, A_{i+1} \cap B_{i+1})$ $(i = 0, 1, \ldots, m)$ to extend the map $p_i g_{i-1}|(\partial\sigma \times I) \cup (\sigma \times 0)$ to a map $g_i|\sigma \times I : (\sigma \times I, \sigma \times 1) \to (A_{i+1}, A_{i+1} \cap B_{i+1})$ or $g_i|\sigma \times I : (\sigma \times I, \sigma \times 1) \to (B_{i+1}, A_{i+1} \cap B_{i+1})$. Thus we obtain a desired map of triads g_i. Then we are done if we let $(P; P', P'') = (P_m \cup P'_m; P_m, P'_m)$, let $f : (X_0; A_0, B_0) \to (P; P', P'')$ be the inclusion and let $g = g_{m-1} : (P; P', P'') \to (X_m; A_m, B_m)$. □

LEMMA 6.5. *Let*

$$(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda) \in \operatorname{ob} \textbf{pro-HPol}_*^T.$$

Then if the inverse systems of pairs $(\boldsymbol{X}_0, \boldsymbol{X}_0 \cap \boldsymbol{X}_1)$ and $(\boldsymbol{X}_1, \boldsymbol{X}_0 \cap \boldsymbol{X}_1)$ are n-connected and m-connected, respectively, and if $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ are 0-connected, then for each $\lambda \in \Lambda$, there exists $\lambda' \geq \lambda$ so that the map of triads $p_{\lambda\lambda'}$ factors through a polyhedral triad $(P; P_0, P_1)$ such that the pairs $(P_0, P_0 \cap P_1)$ and $(P_1, P_0 \cap P_1)$ are n-connected and m-connected, respectively.

PROOF. Without loss of generality, we can assume $n \leq m$ and that all $X_{0\lambda}$ and $X_{1\lambda}$ are connected. Then for each $\lambda \in \Lambda$ we have $\lambda = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_m \leq \lambda_{m+1} = \lambda'$ so that $(p_{\lambda_i\lambda_{i+1}}|X_{0\lambda_{i+1}})_* = 0 : \pi_{m-i}(X_{0\lambda_{i+1}}, X_{0\lambda_{i+1}} \cap X_{1\lambda_{i+1}}) \to \pi_{m-i}(X_{0\lambda_i}, X_{0\lambda_i} \cap X_{1\lambda_i})$ for $i = m, m-1, \ldots, m-n$ and $(p_{\lambda_i\lambda_{i+1}}|X_{1\lambda_{i+1}})_* = 0 : \pi_{m-i}(X_{1\lambda_{i+1}}, X_{0\lambda_{i+1}} \cap X_{1\lambda_{i+1}}) \to \pi_{m-i}(X_{1\lambda_i}, X_{0\lambda_i} \cap X_{1\lambda_i})$ for $i = m, m-1, \ldots, 0$. Then the lemma follows from Lemma 6.4. □

PROOF OF THEOREM 6.3. Let $(X; X_0, X_1)$ be as in the hypothesis, and let $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ be a polyhedral resolution of $(X; X_0, X_1)$. Without loss of generality, we can assume $n \leq m$ and that all $(X_\lambda; X_{0\lambda}, X_{1\lambda})$ are polyhedral triads such that all $X_{0\lambda}$ and $X_{1\lambda}$ are connected. Fix $\lambda \in \Lambda$. Then by Lemma 6.5, there exist $\lambda' \geq \lambda$, a polyhedral triad $(P; P_0, P_1)$ such that the pairs $(P_0, P_0 \cap P_1)$ and $(P_1, P_0 \cap P_1)$ are n-connected and m-connected, respectively, and maps of triads $f : (X_{\lambda'}; X_{0\lambda'}, X_{1\lambda'}) \to (P; P_0, P_1)$ and $g : (P; P_0, P_1) \to (X_\lambda; X_{0\lambda}, X_{1\lambda})$ such that $p_{\lambda\lambda'} = gf$. Then the Blakers-Massey theorem in homotopy theory implies that the inclusion $j : (P_0, P_0 \cap P_1) \hookrightarrow (P, P_1)$ induces the map $j_* : \pi_r(P_0, P_0 \cap P_1) \to \pi_r(P, P_1)$ which is an isomorphism for $1 \leq r \leq n + m - 1$ and an epimorphism for $r = n + m$. Consider the induced commutative diagram in homotopy sets:

$$\begin{array}{ccccc}
\pi_r(X_{0\lambda}, X_{0\lambda} \cap X_{1\lambda}) & \xleftarrow{g_*} & \pi_r(P_0, P_0 \cap P_1) & \xleftarrow{f_*} & \pi_r(X_{0\lambda'}, X_{0\lambda'} \cap X_{1\lambda'}) \\
{\scriptstyle i_{\lambda *}}\downarrow & & {\scriptstyle i_*}\downarrow & & {\scriptstyle i_{\lambda' *}}\downarrow \\
\pi_r(X_\lambda, X_{1\lambda}) & \xleftarrow{g'_*} & \pi_r(P, P_1) & \xleftarrow{f'_*} & \pi_r(X_{\lambda'}, X_{1\lambda'})
\end{array}$$

where the vertical maps are induced by the inclusions. For $1 \leq r \leq n+m-1$, let $h = g_*(i_*)^{-1} f'_* : \pi_r(X_{\lambda'}, X_{1\lambda'}) \to \pi_r(X_{0\lambda}, X_{0\lambda} \cap X_{1\lambda})$. Then h fills the diagonal of the following commutative diagram:

$$\begin{array}{ccc}
\pi_r(X_{0\lambda}, X_{0\lambda} \cap X_{1\lambda}) & \xleftarrow{(p_{\lambda\lambda'}|X_{0\lambda'})_*} & \pi_r(X_{0\lambda'}, X_{0\lambda'} \cap X_{1\lambda'}) \\
{\scriptstyle i_{\lambda *}}\downarrow & & {\scriptstyle i_{\lambda' *}}\downarrow \\
\pi_r(X_\lambda, X_{1\lambda}) & \xleftarrow{(p_{\lambda\lambda'}|X_{\lambda'})_*} & \pi_r(X_{\lambda'}, X_{1\lambda'})
\end{array}$$

Morita's lemma [**6**, Theorem 5, p.113] implies $\boldsymbol{i}_* = (i_{\lambda *}) : \pi_r(\boldsymbol{X}_0, \boldsymbol{X}_0 \cap \boldsymbol{X}_1) \to \pi_r(\boldsymbol{X}, \boldsymbol{X}_1)$ is an isomorphism for $1 \leq r \leq n + m - 1$. Also for $r = n + m$, $i_* : \pi_r(P_0, P_0 \cap P_1) \to \pi_r(P, P_1)$ is an epimorphism, so $\mathrm{Im}((p_{\lambda\lambda'}|X_{\lambda'})_*) \subseteq \mathrm{Im}(i_{\lambda *})$. Then [**6**, Theorem 3, p. 109] implies that $\boldsymbol{i}_* = (i_{\lambda *}) : \pi_{n+m}(\boldsymbol{X}_0, \boldsymbol{X}_0 \cap \boldsymbol{X}_1) \to \pi_{n+m}(\boldsymbol{X}, \boldsymbol{X}_1)$ is an epimorphism. This completes the proof of Theorem 6.3. □

7. Mayer-Vietoris sequences

For each abelian group G, let $\check{H}^r(\ ; G)$ denote the r-th Čech cohomology theory with coefficients in G which is based on the normal open coverings. G will be omitted as long as no confusion occurs. Let $(X; X_0, X_1)$ be a triad of spaces such that X_0 and X_1 are normally embedded closed subspaces of X. Then Theorem 3.9 implies the existence of an $\mathbf{HPol}^T$-expansion $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ of $(X; X_0, X_1)$ such that the induced morphisms $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$, $\boldsymbol{p}|X_i = (p_\lambda|X_i) : X_i \to \boldsymbol{X}_i$, $i = 0, 1$, and $\boldsymbol{p}|X_0 \cap X_1 = (p_\lambda|X_0 \cap X_1) : X_0 \cap X_1 \to \boldsymbol{X}_0 \cap \boldsymbol{X}_1 = (X_{0\lambda} \cap X_{1\lambda}, p_{\lambda\lambda'}|X_{0\lambda'} \cap X_{1\lambda'}, \Lambda)$ are expansions. Then for each $\lambda \in \Lambda$, there is a Mayer-Vietoris sequence of the polyhedral triad $(X_\lambda; X_{0\lambda}, X_{1\lambda})$, which is exact and natural. Hence there is an induced Mayer-Vietoris sequence of

Čech cohomology groups $\mathrm{MV}(X; X_0, X_1)$:

$$\to \check{H}^{r-1}(X_0 \cap X_1) \xrightarrow{\delta} \check{H}^r(X) \to \check{H}^r(X_0) \oplus \check{H}^r(X_1) \to \check{H}^r(X_0 \cap X_1) \to$$

Then [**6**, Lemma 1, p. 129] implies the following:

THEOREM 7.1. *For each triad of spaces $(X; X_0, X_1)$ such that X_0 and X_1 are normally embedded closed subsets of X, the Mayer-Vietoris sequence MV $(X; X_0, X_1)$ of Čech cohomology groups is exact.*

Let $\mathcal{MV}$ denote the category whose objects are triads of spaces $(X; X_0, X_1)$ such that X_0 and X_1 are normally embedded closed subsets of X and whose morphisms $\Phi : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ are homomorphisms of Mayer-Vietoris sequences (see [**1**, p. 8]) from $\mathrm{MV}(X; X_0, X_1)$ to $\mathrm{MV}(Y; Y_0, Y_1)$. Also let $\mathbf{Sh}^T_N$ denote the full subcategory of $\mathbf{Sh}^T$ whose objects are triads of spaces $(X; X_0, X_1)$ such that X_0 and X_1 are normally embedded closed subsets of X. Then we have

THEOREM 7.2. *There exists a contravariant functor $\mathcal{F}$ from $\mathbf{Sh}^T_N$ to $\mathcal{MV}$.*

PROOF. For each $(X; X_0, X_1) \in \mathrm{ob}\,\mathbf{Sh}^T_N$, let $\mathcal{F}$ be the identity on the objects, i.e., $\mathcal{F}(X; X_0, X_1) = (X; X_0, X_1)$ for each $(X; X_0, X_1) \in \mathrm{ob}\,\mathbf{Sh}^T_N$. Let $\varphi \in \mathbf{Sh}^T_N((X; X_0, X_1), (Y; Y_0, Y_1))$ be represented by the morphism $\boldsymbol{\varphi} = (\varphi_\mu) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ where $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ and $\boldsymbol{q} = (q_\mu) : (Y; Y_0, Y_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ are the $\mathbf{HPol}^T$-expansions of $(X; X_0, X_1)$ and $(Y; Y_0, Y_1)$, respectively, such that the induced morphisms $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$, $\boldsymbol{p}|X_i = (p_\lambda|X_i) : X_i \to \boldsymbol{X}_i$ for $i = 0, 1$, $\boldsymbol{p}|X_0 \cap X_1 = (p_\lambda|X_0 \cap X_1) : X_0 \cap X_1 \to \boldsymbol{X}_0 \cap \boldsymbol{X}_1$, $\boldsymbol{q} = (q_\mu) : Y \to \boldsymbol{Y}$, $\boldsymbol{q}|Y_i = (q_\mu|Y_i) : Y_i \to \boldsymbol{Y}_i$ for $i = 0, 1$, and $\boldsymbol{q}|Y_0 \cap Y_1 = (q_\mu|Y_0 \cap Y_1) : Y_0 \cap Y_1 \to \boldsymbol{Y}_0 \cap \boldsymbol{Y}_1$ are all expansions. Then the morphisms induced by $\boldsymbol{\varphi}$, $\boldsymbol{\varphi} = (\varphi_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{\varphi}|X_i = (\varphi_\mu|X_{i\varphi(\mu)}) : \boldsymbol{X}_i \to \boldsymbol{Y}_i$ for $i = 0, 1$, $\boldsymbol{\varphi}|X_0 \cap X_1 = (\varphi_\mu|X_{0\varphi(\mu)} \cap X_{1\varphi(\mu)}) : \boldsymbol{X}_0 \cap \boldsymbol{X}_1 \to \boldsymbol{Y}_0 \cap \boldsymbol{Y}_1$ define the morphisms $\varphi|X \in \mathbf{Sh}(X, Y)$, $\varphi|X_i \in \mathbf{Sh}(X_i, Y_i)$, $i = 0, 1$, and $\varphi|X_0 \cap X_1 \in \mathbf{Sh}(X_0 \cap X_1, Y_0 \cap Y_1)$ which make the following diagram commute for $i = 0, 1$:

$$\begin{array}{ccccc} X_0 \cap X_1 & \xrightarrow{j} & X_i & \xrightarrow{k} & X \\ {\scriptstyle \varphi|X_0\cap X_1}\downarrow & & {\scriptstyle \varphi|X_i}\downarrow & & \downarrow{\scriptstyle \varphi|X} \\ Y_0 \cap Y_1 & \xrightarrow{j'} & Y_i & \xrightarrow{k'} & Y \end{array}$$

where the horizontal maps are the inclusions. Here $\mathbf{Sh}$ denotes the shape category in the sense of [**6**]. Thus we have the following commutative diagram:

$$
\begin{array}{ccccc}
\longrightarrow & \check{H}^{r-1}(X_0 \cap X_1) & \xrightarrow{\delta} & \check{H}^r(X) & \longrightarrow \cdots \\
 & {\scriptstyle (\varphi|X_0\cap X_1)^*}\uparrow & & {\scriptstyle (\varphi|X)^*}\uparrow & \\
\longrightarrow & \check{H}^{r-1}(Y_0 \cap Y_1) & \xrightarrow{\delta} & \check{H}^r(Y) & \longrightarrow \cdots
\end{array}
$$

$$
\begin{array}{ccccc}
\cdots \longrightarrow & \check{H}^r(X_0) \oplus \check{H}^r(X_1) & \longrightarrow & \check{H}^r(X_0 \cap X_1) & \longrightarrow \\
 & {\scriptstyle ((\varphi|X_0)^*,(\varphi|X_1)^*)}\uparrow & & {\scriptstyle (\varphi|X_0\cap X_1)^*}\uparrow & \\
\cdots \longrightarrow & \check{H}^r(Y_0) \oplus \check{H}^r(Y_1) & \longrightarrow & \check{H}^r(Y_0 \cap Y_1) & \longrightarrow
\end{array}
$$

Let $\mathcal{F}(\varphi)$ be the homomorphism from $\mathrm{MV}(Y; Y_0, Y_1)$ to $\mathrm{MV}(X; X_0, X_1)$ which is defined by this diagram. It is easy to show that $\mathcal{F}$ defines a functor. □

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