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Discrete-Time, Linear Periodic Time-Varying System Norm Estimation Using Finite Time Horizon Transfer Operators

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The norm is one of the fundamental concepts of linear algebra and functional analysis. The notion of the norm is often employed in engineering, e.g. in control engineering, where main application is calculating the norm of the transfer function. Unfortunately existing methods are applicable for systems that can be described using Laplace transform, i.e. linear time-invariant (LTI) systems. An operational equivalent of the transfer function for linear time-varying systems is transfer operator. The transfer operator defined for finite time horizon can be described by finite dimensional matrix. Although for infinite time horizon the operator is infinite dimensional. In the paper a method for norm estimation of transfer operator defined on infinite time horizon is proposed. The method is applicable for linear time-varying, discrete-time systems given in general state-space form. The method takes advantage of the properties of the transfer operator norm on a finite time horizon. Theoretical considerations are complemented by numerical examples.

Key words: Norm estimation, Discrete-time systems, Time-varying systems, Non-stationary systems

Estimacija norme diskretnih, periodički vremenski promjenljivih, linearnih sustava primjenom prijenosnog operatora s konačnim vremenskih horizontom. Norma je jedan od osnovnih koncepata linearne algebre i funkcionalne analize. Pojam norme se često koristi kod inžinjera, npr. kod upravljanja, gdje je jedna od glavnih aplikacija računanje norma prijenosne funkcije. Nažalost postojeće metode su primjenjive samo na sustave koji se mogu opisati koristeći Laplaceovu transformaciju, tj. linearne vremenski nepromjenjive sustave. Ekvivalent prijenosnoj funkciji za linearne vremenski promjenjive sustave je prijenosni operator. Prijenosni operator definiran za konačni vremenski horizont može se opisati konačno dimenzionalnom matricom. Iako je za beskonačni vremenski horizont operator beskonačno dimenzionalan. U radu je predložena metoda za estimaciju norme prijenosnog operatora definiranog na beskonačnom vremenskom horizontu. Metoda je primjenjiva na linearne vremenski promjenjive diskretne sustave zadane u obliku prostora stanja. Metoda koristi svojstva norme prijenosnog operatora za konačni vremenski horizont. Teoretska promišljanja nadopunjena su numeričkim primjerima.

Ključne riječi: estimacija norme, diskretni sustavi, vremenski promjenjivi sustavi, nestacionarni sustavi

1 INTRODUCTION

Linear time-varying approach is of relevant interest in adaptive control, multi-model design with improved transient performances and switching operations in piecewise affine systems. In order to describe the dynamics of time-varying discrete-time systems, one can use difference equations with time-dependent coefficients or a generalized description employing state equations with time-dependent matrices in following form:

$$\begin{aligned} &x(k+1) = A(k)x(k) + B(k)v(k) \\ &y\left(k\right) = C\left(k\right)x\left(k\right) + D\left(k\right)v\left(k\right) \\ &x\left(k_0\right) = x_0 \end{aligned} \tag{1}$$

where $x\left(k\right) \in \mathbb{R}^n$ is nominal state, $v\left(k\right) \in \mathbb{R}^m$ is the nominal control, $y\left(k\right) \in \mathbb{R}^p$ is the nominal output and $A\left(k\right) \in \mathbb{R}^{n\times n}$, $B\left(k\right) \in \mathbb{R}^{n\times m}$, $C\left(k\right) \in \mathbb{R}^{p\times n}$, $D\left(k\right) \in \mathbb{R}^{p\times m}$ are system matrices, $k = k_0, k_0+1, ..., k_0+N$ and N is length of the time horizon. For infinite time horizon $N = \infty$.

An LTV system can be equivalently described in terms of the matrix operators. There are two different approaches: one based on block diagonal operators [1] and the other based on a lower triangular system matrix [2]. Both approaches lead to an operator-based description of the system and a function which takes the role of a transfer function for time-varying systems. This function has many properties analogous to those of transfer functions of linear time-invariant (LTI) systems. In some cases, this allows one to apply to linear time-varying (LTV) systems techniques which have formerly been restricted to LTI systems. Some machinery and results of robust control in particular would be available for LTV systems.

2 OPERATORS NOTATION

In order to describe the dynamics of time-varying discrete-time systems, one can employ state space equations with time-dependent matrices given by (1). Alternatively, the model may be described by means of operators. Equations (1) can be converted into following operators form:

$$\hat{y} = \hat{C}\hat{N}x_0 + \left(\hat{C}\hat{L}\hat{B} + \hat{D}\right)\hat{v} = \hat{C}\hat{N}x_0 + \hat{T}\hat{v} \quad (2)$$

In order that the system (2) be equivalent to the system (1), operators $\hat{T} = \hat{C}\hat{L}\hat{B} + \hat{D}$ and $\hat{C}\hat{N}$ must be defined in one of the two equivalent notations: either an evolutionary one, where operators are written by means of sums and products [3]:

$$\begin{aligned} y(k) &= \left(\hat{C}\hat{N}x_0\right)(k) + \left(\hat{C}\hat{L}\hat{B}\hat{v}\right)(k) + D(k)\,v(k) \\ &= C(k)\,\varphi_{k_0}^{k-1}x_0 + D(k)\,v(k) \\ &+ C(k)\left(\textstyle\sum_{i=k_0}^{k-2}\varphi_{i+1}^{k-1}B(i)\,v(i) + B(k-1)\,v(k-1)\right) \end{aligned} \quad (3)$$

where $\varphi_i^k = A(k)\,A(k-1)\ldots A(i)$, or a matrix-based one, where each of the operators can be presented in terms of matrices. In order to analyze the stability of the system, one has to know operators $\hat{T}$ and $\hat{N}$ which can be expressed with the help of the following matrix operators:

$$\hat{L} = \begin{bmatrix} I & 0 & \cdots & 0 \\ \varphi_{k_0+1}^{k_0+1} & I & 0 & \vdots \\ \vdots & \ddots & I & 0 \\ \varphi_{k_0+1}^{k_0+N-1} & \cdots & \varphi_{k_0+N-1}^{k_0+N-1} & I \end{bmatrix} \quad (4)$$

$$\hat{N} = \begin{bmatrix} \varphi_{k_0}^{k_0} \\ \vdots \\ \varphi_{k_0}^{k_0+N-1} \end{bmatrix} \quad (5)$$

$$\hat{B} = \begin{bmatrix} B(k_0) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B(k_0+N-1) \end{bmatrix} \quad (6)$$

$$\hat{C} = \begin{bmatrix} C(k_0) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C(k_0+N-1) \end{bmatrix} \quad (7)$$

Operator $\hat{N}$ can be neglected when initial conditions are zero. Following sequences: state $\hat{x}$, output $\hat{y}$ and input $\hat{v}$ are constructed from state $x(k)$, output $y(k)$ and input $v(k)$ signals rewritten in following block column vector form:

$$\hat{x} = \left[x^T(k_0+1)\cdots x^T(k_0+N)\right]^T \quad (8)$$

$$\hat{y} = \left[y^T(k_0+1)\cdots y^T(k_0+N)\right]^T \quad (9)$$

$$\hat{v} = \left[v^T(k_0+1)\cdots v^T(k_0+N)\right]^T \quad (10)$$

The input/output operator $\hat{T}$ can be alternatively defined also using a set of impulse responses of a time-varying system taken at different times, e.g. for SISO system it may be written:

$$\hat{T} = \begin{bmatrix} h(k_0,k_0) & 0 & \cdots & 0 \\ h(k_0+1,k_0) & h(k_0+1,k_0+1) & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h(k_0+N-1,k_0) & \cdots & \cdots & h(k_0+N-1,k_0+N-1) \end{bmatrix} \quad (11)$$

where $h(k_1,k_0)$ is the response of the system to the Kronecker delta $\delta(k-k_0)$ at time k_1 (after k_1–k_0 samples). In the case of a nonzero input-output delay operator, $\hat{D} = 0$ and all diagonal entries of $\hat{T}$ are equal to zero.

For further considerations in the paper following definitions of norms for sequences and operators are used. The norm of a sequence in the Hilbert-space is understood as Euclidean norm:

$$\|\hat{v}\| = \|\hat{v}\|_2 = \sqrt{\langle\hat{v},\hat{v}\rangle} = \sqrt{\sum_k v^T(k)\,v(k)} = \sqrt{\hat{v}^T\hat{v}} \quad (12)$$

The ∞-norm of a sequence in the bounded sequences space is understood as:

$$\|\hat{v}\|_\infty = \max_k\left(|v(k)|\right) \quad (13)$$

Norms of operators are defined in following way:

$$\left\|\hat{T}\right\| = \left\|\hat{T}\right\|_2 = \sup_{\hat{v}\neq 0}\frac{\left\|\hat{T}\hat{v}\right\|_2}{\|\hat{v}\|_2} \quad (14)$$

$$\left\|\hat{T}\right\|_\infty = \sup_{\hat{v}\neq 0}\frac{\left\|\hat{T}\hat{v}\right\|_\infty}{\|\hat{v}\|_\infty} \quad (15)$$

For systems defined on finite time horizon all operators are represented by finite dimensional matrices and signals by finite dimensional vectors. Moreover the input-output operator is a compact, Hilbert-Schmidt operator from *l2* into *l2* and actually maps bounded signals $v \in \mathcal{M} = l_2[k_0, k_0+N]$ into the signals $y \in$.

3 COMPUTATION THE NORM OF THE TIME-VARYING SYSTEM

Stability and performance criteria for analysis and robust control design of linear systems, are often expressed by norms of appropriately defined transfer functions or transfer operators, especially for time varying systems. Norms of the linear time-invariant systems defined on infinite time horizon can be easily computed using algorithms described in [4], [5]. The algorithms are also implemented in Matlab Control Toolbox [6]. They needs only conversion of the system operator into state-space description. Although many methods for computing norms for linear time-invariant systems [4], [7], [8] which are essential in a computer aided control system design [9] there are very difficult to find methods applicable for linear time-varying systems.

Very important and difficult is the problem of norm estimation for wider class of time-varying systems. Generally following classes can be distinguished for matrix A (similar conditions holds for other system matrices B, C, D):

1. Periodic coefficient matrices. System matrices can be computed for all $k \in \mathbb{Z}$ from condition:

$$A_P\left(k+iP\right) = A_P\left(k\right) \tag{16}$$

   where $0 \leq k < P,\ i \in \mathbb{Z}$ and *P* is discrete period of the parameter variability.

2. Almost-periodic coefficient matrices. The coefficient matrix $A\left(k\right)$ is said to be almost periodic if

$$\lim_{k\to\infty} A\left(k\right) = A_P\left(k\right) \tag{17}$$

   where $A_P\left(k\right)$ is periodic with period *P*, see eq. (16).

3. Almost-constant coefficient matrices. The coefficient matrix $A\left(k\right)$ is said to be almost constant if

$$\lim_{k\to\infty} A\left(k\right) = A \tag{18}$$

   where A is constant matrix.

   General time-varying system.

4. System defined on finite time horizon. The system matrices are defined only on some given bounded time horizon

$$A\left(k\right),\ k_0 \leq k < k_0+N \tag{19}$$

There is no assumptions about past and future system behaviour.

The theory of linear discrete-time periodic systems has received a lot of attention in the last 25 years [10], [11]. Most theoretical results are based on two lifting techniques which reduce the problem for the periodic system to an equivalent problem for a linear time-invariant (LTI) system of increased dimensions [12], [13], [14]. The lifting approach either involves forming products of up to K matrices, where K is the period of the system [12] or leads to a large order standard system representation with sparse and highly structured matrices [13]. It has been shown in [15] that the lifting approach is based on isometric isomorphism of l_2 onto a new space where the operators are Toeplitz. Computational and numerical problems associated with these techniques have been studied by [16], [17]. More useful information about these techniques can be found in the overview article [11]. These frequency-domain methods for the analysis of periodic systems rely on the transfer function matrix of the associated lifted systems [12], [13]. Using the method computation of frequency responses can be done by computing first the corresponding transfer function matrix and then evaluating the frequency response using the resulting rational matrix. A method to compute the transfer function matrix can be devised along the lines of the pole/zero approach [16]. Alternatively, the frequency response can be computed by exploiting the sparse structure of the lifted representation of the periodic system [17].

Norm of transfer operator defined on infinite time horizon can be also computed for periodic linear time-varying systems employing lifting technique. The paper [14] is an overview and comparison of techniques which allows to rewrite time-varying systems using time-invariant representation with increased but finite dimensions. Norm of the transfer operator for such system can be computed in similar way as for linear time-invariant systems. More description for the lifting technique for periodic time-varying systems can be found in [12], [13], [16], [17], [18].

Nevertheless norm of systems 2-4 cannot be easily computed. The norm of transfer operator for systems 2-4 can be estimated using general operator theory [19], [20], [21], [22], [23] or the technique based on parameterised functional minimization. The main idea is based on the following general result given in [24].

Theorem 1 *Let $\mathcal{M}$, be real Hilbert spaces, $\hat{T} \in \mathcal{L}\left(\mathcal{M}, \right)$, is bounded transfer operator for causal system, which belongs to the lower triangular block operators space, $\hat{C}\hat{N} \in$, $\gamma \in (0, \infty)$ and $J\left(\hat{v}\right)$ be a functional defined on $\mathcal{M}$ and given by*

$$J(\hat{v}) = \left\|\hat{T}\hat{v} + \hat{C}\hat{N}\right\|^2 - \gamma^2 \left\|\hat{v}\right\|_{\mathcal{M}}^2 \tag{20}$$

(a) $\left\|\hat{T}\right\| < \gamma$ if and only if there exists β >0, such that

$$\left\|\hat{T}\hat{v}\right\|^2 - \gamma^2 \left\|\hat{v}\right\|^2_{\mathcal{M}} \leq -\beta \left\|\hat{v}\right\|^2_{\mathcal{M}} \forall \hat{v} \in \mathcal{M} \tag{21}$$

Consequently, if $\left\|\hat{T}\right\| < \gamma$*, then* (20) *always achieves a unique finite maximum over*$\mathcal{M}$*.*

(b) If $\left\|\hat{T}\right\| > \gamma$ *then* (20) *does not achieve a finite maximum over* $\mathcal{M}$*, i.e.* $\sup\limits_{\hat{v}\in\mathcal{M}} J(\hat{v}) = +\infty$.

It mean that $\left\|\hat{T}\right\| = \inf\gamma$ over all γ such that the maximization of (20) has a finite solution. The required value of γ can be found with arbitrary accuracy, e.g. by means of the bisection method. Equivalence between the maximization of the functional (20) and the existence of a solution to the corresponding Riccati difference equations can be exploited.

Estimation of the operator norm using the method of parameterised functional minimization in general can takes large computational power.

In order to make computationally efficient norm estimation, following approach based of finite-time horizon norm is proposed.

Definition 1 *Amplification energy factor* k_e *for system with zero initial condition* $x_0 = 0$ *is given in following way*

$$k_e = \frac{\|\hat{y}\|}{\|\hat{v}\|} = \sqrt{\frac{\hat{y}^{\mathrm{T}}\hat{y}}{\hat{v}^{\mathrm{T}}\hat{v}}} = \sqrt{\frac{\sum_{i=1}^{N} y^2(i)}{\sum_{i=1}^{N} v^2(i)}} \tag{22}$$

For systems unstable in the input-output sense output energy grows unboundedly for bounded input signals, i.e. $\sup\limits_{\hat{v}\neq 0}(k_e) = \infty$*. It implies infinite value of the norm of transfer operator, i.e.*

$$\left\|\hat{T}\right\| = \sup_{\hat{v}\neq 0}(k_e) \tag{23}$$

where the norm $\left\|\hat{T}\right\| \to \infty$*.*

For systems stable in the input-output sense output energy is bonded for bounded input signals, i.e. $0 \leq k_e < \infty$*. It implies finite value of the norm of transfer operator* $\left\|\hat{T}\right\|$*.*

Let us assume that a system defined on infinite time horizon will be considered as a system defined on finite time horizon with length *N*. The norm of transfer operator of the system defined on finite time horizon *N* be denoted in following way:

$$\left\|\hat{T}_{[N]}\right\| \tag{24}$$

where

$$\forall_{N\in\mathbb{Z}} \quad \left\|\hat{T}_{[N-1]}\right\| \leq \left\|\hat{T}_{[N]}\right\| \tag{25}$$

If the norm of transfer operator defined on infinite time horizon is finite $\left\|\hat{T}\right\| = c$ then there exist a limit c such that:

$$\lim_{N\to\infty} \left\|\hat{T}_{[N]}\right\| = c \tag{26}$$

Thus for large enough lengths of the time horizon it may be concluded that finite time horizon norm is an approximation of the infinite time horizon norm, i.e.:

$$\forall_{N\geq N_0} \left\|\hat{T}_{[N]}\right\| \cong \left\|\hat{T}\right\| \tag{27}$$

Relative approximation error can be expressed by following equation:

$$\delta\left(\hat{T}, N\right) = \left|\frac{\left\|\hat{T}_{[N]}\right\|}{\left\|\hat{T}\right\|} - 1\right| \tag{28}$$

Although it is impossible to find simple relation between the relative error δ and the length of the time horizon *N* for general time-varying system $\hat{T}$, we show that the method is very simple and very efficient alternative for discrete-time, time-varying systems norm estimation.

4 NUMERICAL ANALYSIS FOR PERIODIC TIME-VARYING SYSTEM

The system under consideration is special case of the linear time-varying system whereas $A(k)$ is the time-varying system matrix with invariant eigenvalues. The system is characterized by constant (time-invariant) eigenvalues of the system matrix despite changes in its entries. This idea is borrowed from [25], [1]. The additional parameter ε allows changes of the system with a degree of non-stationarity as well as the pole location. Eigenvalues of matrix $A(k)$ are inside the unitary circle, but can be either stable or unstable with respect to switching in the structure of the system. The deciding factor is the switching interval defined by the parameter ε. System matrices (1) are the following:

$$\begin{gathered} A(k) = A_\kappa, \quad B(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \\ C(k) = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D(k) = 0 \end{gathered} \tag{29}$$

where

$$A_0 = \begin{bmatrix} 2 & 1.2 \\ -2 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -2 \\ 1.2 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1.2 \\ -2 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & -2 \\ 1.2 & -1 \end{bmatrix}, \kappa = floor\left(rem\left(\tfrac{k}{\varepsilon}, 4\right)\right) \tag{30}$$

Variable κ denotes rounding towards negative infinity (floor) of the remanent (signed remainder of κ/ε after division by 4). Eigenvalues of the matrix $A(k)$ are independent of the parameter ε and equal to 0.5 ± 0.3873i for all k.

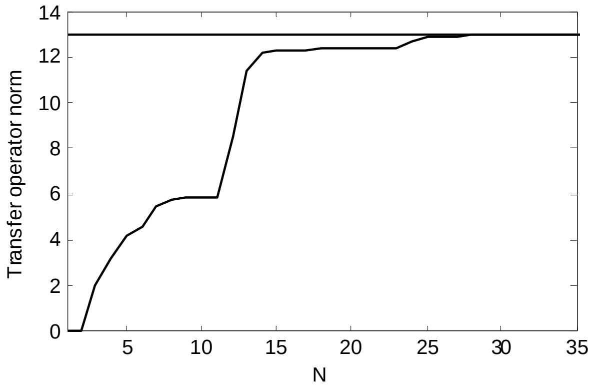

Fig. 1. Norm of transfer operator for finite time horizon discrete switching system (29)-(30) *with* $\varepsilon = 5$ *vs. the length of the time horizon* N

In fact value of the parameter ε significantly changes properties of the system. Small values $\varepsilon < 2.8$ implies unstable character of the system whereas large values results in stable, switching system. Figure 1 shows values of the transfer operator norm $\left\|\hat{T}_{[N]}\right\|$ vs. length of the time horizon N for $\varepsilon = 5$. Value estimated using lifting techniques is equal to $\left\|\hat{T}\right\| = 12.9849$ and depicted by dotted line. As can be seen from fig. 1 estimated norm fast reach neighbourhood of the real value. It takes only about 27 time steps.

Relative error for the same system computed for the length of the time horizons up to 500 is depicted in fig. 2. From practical point of view relative error for norm estimation below 10^{-2} is in most cases sufficient, in this case it takes only 27 time steps what is relatively fast, even for second order system but with variability period of $4\varepsilon = 20$ time steps.

Figure 3 shows values of the transfer operator norm $\left\|\hat{T}_{[N]}\right\|$ vs. length of the time horizon N for the discrete-time switching system with $\varepsilon = 40$. Value estimated using

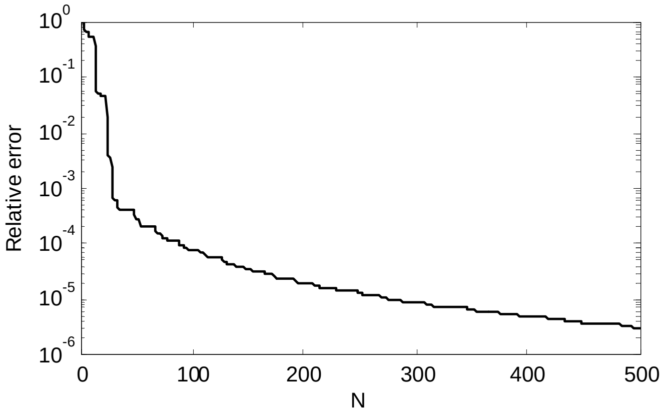

Fig. 2. Relative error of the transfer operator norm computed on finite time horizon for discrete switching system (29)-(30) *with* $\varepsilon = 5$ *vs. the length of the time horizon* N.

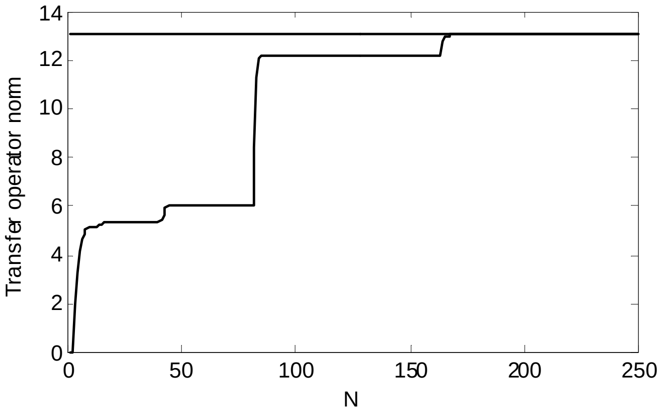

Fig. 3. Norm of transfer operator for finite time horizon discrete switching system (29)-(30) *with* $\varepsilon = 40$ *vs. the length of the time horizon* N.

lifting techniques is equal to $\left\|\hat{T}\right\| = 13.053$ and depicted by dotted line. Now the variability period of the systems is equal to $4\varepsilon = 160$ samples. As can be seen from fig. 1 estimated norm reach fast neighbourhood of the real value, just after one full period. Relative error for the length of the time horizons up to 250 is depicted in fig. 4. After the first full period the error decrease very fast achieving value about 10^{-15} resulting from numerical accuracy.

Figure 5 shows values of the transfer operator Euclidean norm $\left\|\hat{T}_{[N]}\right\|$ vs. length of the time horizon N and the parameter ε for the discrete-time switching system. The norm need a short number of samples to achieve stable level for stable configurations $\varepsilon > 3$ and very fast increase for unstable configurations. Values of the estimated norm are limited on the fig. 5 to maximal level of 40. Similar 3D diagram depicted in fig. 6 shows values of the transfer

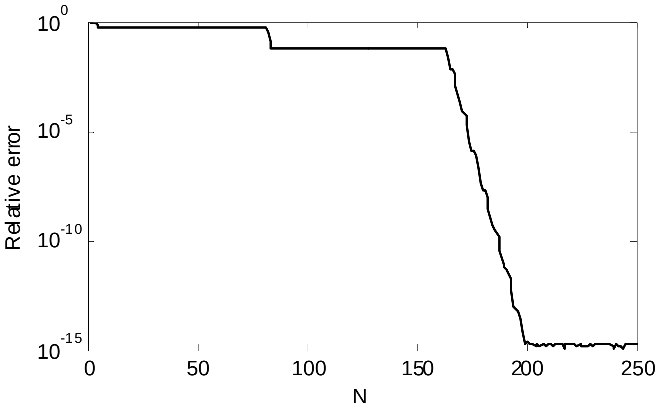

Fig. 4. Relative error of the transfer operator norm computed on finite time horizon for discrete switching system (29)-(30) *with* $\varepsilon = 40$ *vs. the length of the time horizon* N.

operator ∞-norm $\left\|\hat{T}_{[N]}\right\|_\infty$ vs. length of the time horizon N and the parameter ε for the discrete-time switching system. Generally the values of the norm are higher, but the tendencies are close together.

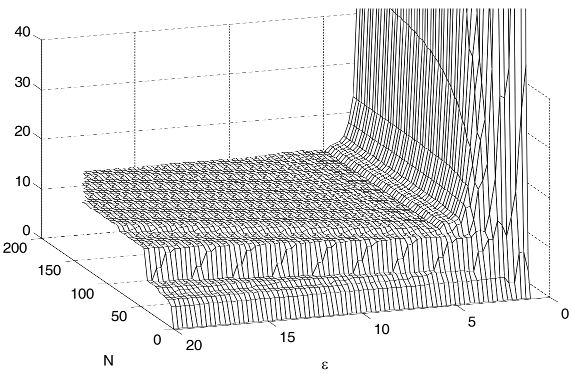

Fig. 5. Euclidean norm of transfer operator for finite time horizon discrete switching system (29)-(30) *vs. the length of the time horizon* N *and the parameter* ε .

Table 1. Comparison between computation time for system norm estimation.

N	$\varepsilon = 5$		$\varepsilon = 40$	
	$t_{2\text{–}norm}$(s)	$t_{inf\text{–}norm}$(s)	$t_{2\text{–}norm}$(s)	$t_{inf\text{–}norm}$(s)
50	$3,2{\cdot}10^{-4}$	$1{\cdot}10^{-5}$	$3,6{\cdot}10^{-4}$	$1{\cdot}10^{-5}$
100	$0,002$	$5{\cdot}10^{-5}$	$0,002$	$5{\cdot}10^{-5}$
200	$0,011$	$2{\cdot}10^{-4}$	$0,012$	$2{\cdot}10^{-4}$
500	$0,12$	$0,003$	$0,12$	$0,003$
Lifting	$0,003$	$0,09$	$0,63$	$23,8$

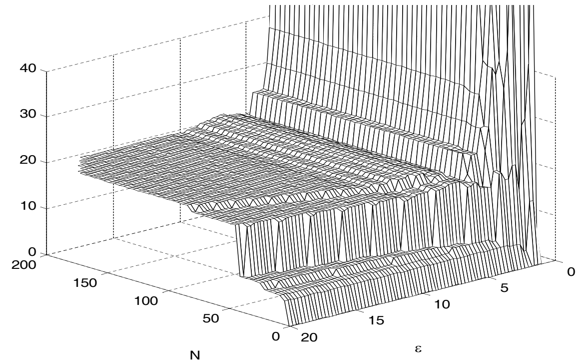

Fig. 6. ∞*-norm of the transfer operator for finite time horizon discrete switching system* (29)-(30) *vs. the length of the time horizon* N *and the parameter* ε .

Computation times for two different norms and two different values of parameter ε are collected in table 1. In the last row are collected computation times for norm estimation using lifting technique. For small variability periods $\varepsilon = 5$ lifting technique requires similar computation time to proposed estimation technique based on running time horizon. Norm estimation using lifting technique for systems with large variability period $\varepsilon = 40$ is much more expensive. Especially much more larger computational time is required for estimation of the $\left\|\hat{T}\right\|_\infty$ norm. Main advantage of the proposed method based on running time horizon is direct applicability for systems with positive real values of the parameter ε. Lifting technique can be applied only for systems with natural numbers of the variability coefficient.

5 CONCLUSION

In the paper a novel approach for the estimation of the operator norm. Particularly infinite dimensional transfer operator norm of dynamical discrete-time, periodical time-varying stable systems can be estimated using block matrix operator notation for transfer operator defined on finite time horizon. The minimal length of the time horizon required for computations is dependent both on the dominant time constant of the system and the variability period of the system matrices. In the considered examples the variability period was higher than the dominant time constant (about 6-10 time steps). Thus the dominant factor is variability period, and the time horizon required to estimate the infinite dimensional operator norm with relative error below 0.001 is equal to 1-1.5 full system periods. One period for system with large $\varepsilon = 40$ - variability period $4\varepsilon = 160$ is about 20 times bigger than the dominant time constant and about 30 time steps for system with smaller $\varepsilon = 5$

with variability period $4\varepsilon = 20$ close to the dominant time constant. Estimates of norm were compared with selected values of system norms computed using lifting techniques for periodic time-varying systems. Both techniques follows to almost identical results. The proposed method based on finite time horizon estimation can be easily applied for systems with fractional values of the switching parameter ε, i.e. for systems with fractional variability periods. Moreover the method can be applied for other linear time-varying systems. i.e. almost periodic and almost constant systems. During the numerical computations it was noticed the method is more computationally efficient, especially for relative error $\delta > 0.001$.

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