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Solving the four index fully fuzzy transportation problem

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Abstract. In this paper, we will solve the four index fully fuzzy transportation problem ($FFTP_4$) with some adapted classical methods. All problem's data will be presented as fuzzy numbers. In order to defuzificate these data, we will use the ranking function procedure. Our method to solve the $FFTP_4$ composed of two phases; in the first one, we will use an adaptation of well-known algorithms to find an initial feasible solution, which are the least cost, Russell's approximation and Vogel's approximation methods. In the second phase, we will test the optimality of the initial solution, if it is not optimal, we will improve it. A numerical analysis of the proposed methods is performed by solving different examples of different sizes; it is determined that they are stable, robust, and efficient. A proper comparative study between the adapted methods identifies the suitable method for solving $FFTP_4$.

Keywords: fuzzy arithmetic, fuzzy transportation problem, linear programming, multi–index problem

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1. Introduction

In the current dynamic market, companies must find the most efficient means of sending products to customers in the quantities requested as soon as possible at the lowest cost; this is the well-known transportation problem (TP). The classical transportation problem has been widely studied [20] and then generalized to more than two indexes [18, 26, 27]. Based on its typology [18], TP can be classified into four groups: 2-index, 3-index, 4-index, and n-index. Previously, many efficient algorithms have been developed to identify the optimal total cost when the demands, offers, and unitary costs are known.

In real-world applications, there are many different situations where classical methods for solving TP are no longer suitable because of the uncertainties of one or more parameters that cannot be accurately determined; this occurs owing to the imprecise tool of measure, lost and missing data, and even calculation errors. To overcome this problem, problem's data are presented with fuzzy numbers, for the purpose of modeling the uncertainties within the data, thus fuzzy mathematics becomes mandatory in solving this type of problems.

The theoretical foundations of fuzzy logic were established in the early 1965 by L- Zadeh [24]. Fuzzy logic is based on the association of the notion of "fuzzy subset" and "theory of possibilities". The world of reasoning in fuzzy logic is more intuitive than in binary logic; it allows designers to better understand natural phenomena and to model them using the definition of rules and membership functions in sets called "fuzzy sets". A fuzzy transportation problem (FTP) is a problem where its data (e.g., costs and or offers and requests) are fuzzy quantities. In 1970, Bellman and Zadeh [2] first introduced the concept of decision-making in a fuzzy

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environment. Eight years later, Zimmermann [25] showed that fuzzy linear programming algorithms were effective. In 1984, Chanas, Kolodziejkzy, and Machaj [3] presented a fuzzy linear programming model to solve the fuzzy transportation problem. Chanas and Kuchta (1996)[4] introduced a new concept for solving a transportation problem with fuzzy numbers data.

In 2004, Liu and Kao [11] described a method for solving FTP using the extension principle. Nagoor Gani and Abdul Razak (2006) [16] obtained a fuzzy solution for the two-stage cost minimizing of FTP in which availabilities and demands are trapezoidal fuzzy numbers. Pandian and Natarajan (2010) [17] proposed a fuzzy zero point method for finding an optimal solution for FTP, where all parameters are trapezoidal fuzzy numbers. Kumar and Kauar (2011) [7] proposed a new method for solving FTP by assuming that a decision maker was uncertain about the exact values of the transportation cost, availability, and demand of the product. Kumar and Murugesan (2012) [9] provided an optimal solution for FTP using triangular fuzzy numbers using a modified revised simplex method. In [10], P. Kumari performed a comparative study on the optimality of the initial solutions obtained by different available methods for the two-index fuzzy transportation problem. In 2020 [19], Jayanta Pratihar et al. proposed an algorithm that was based on Vogel's approximation method to solve the fuzzy transportation problem.

The two-index fuzzy transportation problem has been rigorously studied using fuzzy triangular numbers in [1, 9, 15], intuitionistic fuzzy numbers in [5, 12, 13, 23], and fuzzy trapezoidal numbers in [7, 10, 14, 16, 17, 21, 22]. In addition, in 2019, Kumar et al. [8] used Pythagorean fuzzy numbers. However the efficiency of these algorithms was not tested for the multi-index fuzzy transportation problem. Authors in [6] proposed an adaptation of R. Zitouni's algorithm [27] for solving the four index fuzzy transportation problem.

In this paper, we will present three adapted algorithms to solve the four index fully fuzzy transportation problem ($FFTP_4$) using both trapezoidal and triangular fuzzy numbers. Using the ranking function of fuzzy availabilities, fuzzy requests, fuzzy means of transportation, fuzzy qualities of the goods, and fuzzy costs, we adapt the least cost cell, Russell's approximation, and Vogel's approximation methods in the fuzzy context with four indexes to obtain the initial basic feasible solution. Of note, FLC4, FRAM4, and FVAM4 are the adapted methods, respectively. The degeneracy case is handled by the detective method to complete the base of the initial solution. Finally, we will improve this solution using [27] algorithm's second phase to obtain the optimal solution of $FFTP_4$. In the numerical simulation, we will present a comparative study between FLC4, FRAM4, and FVAM4 in various $FFTP_4$ examples of different sizes (from 8×16 to 113×630000). Based on these results, we will choose a suitable algorithm that provides the initial solution that is close to the optimum in a less number of iterations and time.

This paper is organized as follows. Section 2 contains a brief introduction to fuzzy logic, its application and some fuzzy preliminaries. In Section 3, we introduce an economic interpretation of the $FFTP_4$ as well as its associated mathematical formulation. In Section 4, we present the different descriptions of the adapted algorithms. Numerical implementation and a comparative study between FLC4, FRAM4 and FVAM4 are conducted in Section 5. Finally, Section 6 is the conclusion.

2. Introduction to fuzzy mathematics

2.1. Fuzzy logic

The fuzzy logic is an extension of Boolean logic, it was created in 1965 by L. Zadeh [24] in order to represent mathematically the imprecision, inaccuracies and uncertainties of systems and natural phenomena, by introducing the degree of truth in the verification of a condition; which means a condition can be in state between 0 and 1 (not totally true and not totally false but true and false with a degree), like the notion of beauty which is subject to several discussion. Fuzzy logic uses an imprecise but very descriptive language to deal with input data more like a human operator.

2.2. Fuzzy logic applications

During the 90s, fuzzy logic becomes a fashion, several researches are published about its applications in various fields (decision making, diagnosis, database, optimization, fuzzy systems control, maintenance of rotating machines, ..., etc.), where no deterministic model exists or is not practically implementable, as well as in situations where the imprecision of the data makes control by classical methods impossible.

2.3. Preliminaries

2.3.1. Fuzzy set

L. Zadeh defines a fuzzy set as follows: "a fuzzy set is a class with a continuum of membership grades". Thus, a fuzzy set $\tilde{A}$ (fuzzy subset of universe of discourse U) is characterized by a membership function. It is defined as a set of couples $(x, \mu_{\tilde{A}}(x))$, where x is an element of the universe of discourse U, and $\mu_{\tilde{A}}$ is the membership function.

2.3.2. Membership function

Instead of imposing the membership of an element on "true" or "false" sets, as in the traditional binary logic, fuzzy logic allows degrees of membership. The degree of membership of a fuzzy set is materialized by a number between 0 and 1. Let U be a set; then, the membership function is defined as $\mu : U \to [0, 1]$. The exact value of the membership function linked to the value of the variable is called the "membership factor". Of note, this membership function is the equivalent of the characteristic function of a classical set.

2.4. Fuzzy numbers

A fuzzy number $\tilde{A}$ is a fuzzy subset of the set of real numbers $\mathbb{R}$ with a membership function $\mu_{\tilde{A}}$.

2.4.1. Trapezoidal fuzzy number

A fuzzy number $\tilde{A} = (a, b, c, d)$ with $a \leq b \leq c \leq d$ is called a trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } b \leq x \leq c, \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d, \\ 0 & \text{if } d < x. \end{cases} \quad (1)$$

The function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$ is continuous, strictly increasing on $[a, b]$ constant on $[b, c]$ and strictly decreasing on $[c, d]$. It is presented by a trapezoidal shape.

2.4.2. Triangular fuzzy number

Triangular fuzzy number is a particular case of trapezoidal fuzzy number, we get it only if $b = c$ in the representation (a, b, c, d). Its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ \frac{d-x}{d-b} & \text{if } b \leq x \leq d, \\ 0 & \text{if } d < x. \end{cases} \tag{2}$$

The function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$ is continuous, strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$. It is presented by a triangle shape.

2.5. Ranking function

Let $F(\mathbb{R})$ be the set of fuzzy numbers. The ranking function is a defuzification tool of fuzzy numbers to crisp numbers $\Re : F(\mathbb{R}) \to \mathbb{R}$. It is used to compare fuzzy numbers.

- if $F(\mathbb{R})$ represents a set of trapezoidal fuzzy numbers,

$$\Re(\tilde{A}) = \frac{a + b + c + d}{4}, \quad \tilde{A} = (a, b, c, d), \tag{3}$$

- if $F(\mathbb{R})$ represents a set of triangular fuzzy numbers,

$$\Re(\tilde{A}) = \frac{a + 2b + d}{4}, \quad \tilde{A} = (a, b, d). \tag{4}$$

For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we have

$$\tilde{A} \leq_{\Re} \tilde{B} \Longleftrightarrow \Re(\tilde{A}) \leq \Re(\tilde{B}), \tag{5}$$

$$\tilde{A} \geq_{\Re} \tilde{B} \Longleftrightarrow \Re(\tilde{A}) \geq \Re(\tilde{B}), \tag{6}$$

$$\tilde{A} =_{\Re} \tilde{B} \Longleftrightarrow \Re(\tilde{A}) = \Re(\tilde{B}). \tag{7}$$

2.6. Arithmetic operations

Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two fuzzy trapezoidal numbers:

Addition: $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$.

Subtraction: $\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3, a_4) \ominus (b_1, b_2, b_3, b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$.

Multiplication:

- When $\Re(\tilde{A}) > 0$,

$$\tilde{A} \otimes \tilde{B} = \left(a_1 \times \Re(\tilde{B}), a_2 \times \Re(\tilde{B}), a_3 \times \Re(\tilde{B}), a_4 \times \Re(\tilde{B})\right). \tag{8}$$

- When $\Re(\tilde{A}) < 0$,

$$\tilde{A} \otimes \tilde{B} = \left(a_4 \times \Re(\tilde{B}), a_3 \times \Re(\tilde{B}), a_2 \times \Re(\tilde{B}), a_1 \times \Re(\tilde{B})\right). \tag{9}$$

2.7. Properties

- The addition of fuzzy numbers is associative and commutative :

$$\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A} \text{ and } \tilde{A} \oplus (\tilde{B} \oplus \tilde{C}) = (\tilde{A} \oplus \tilde{B}) \oplus \tilde{C} \quad \forall \tilde{A}, \tilde{B}, \tilde{C} \in F(\mathbb{R}).$$

- No element $\tilde{A} \in F(\mathbb{R})$ has an opposite in $F(\mathbb{R})$.
- The multiplication of a fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ by a real number $\alpha > 0$ is defined by: $\alpha\tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4)$.
- The multiplication of a fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ by a real number $\alpha < 0$ is defined by: $\alpha\tilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1)$.
- $(\alpha + \beta)\ \tilde{A} = \alpha\ \tilde{A} \oplus \beta\ \tilde{A} \quad \forall \alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$ and $\forall\ \tilde{A} \in F(\mathbb{R})$.
- $\alpha(\tilde{A} \oplus \tilde{B}) = \alpha\tilde{A} \oplus \alpha\tilde{B} \quad \forall \alpha \in \mathbb{R}$ and $\forall\ \tilde{A},\ \tilde{B} \in F(\mathbb{R})$.

3. Four index fuzzy transportation problem

Let:

- $A_1, \ldots, A_m$, m be the origins of availabilities $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m$, respectively.
- $B_1, \ldots, B_n$, n be the destinations of demands $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$, respectively.
- $S_1, \ldots, S_p, p$ be the means of transportation chosen depending on reserved charges $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p$ respectively.
- $H_1, \ldots, H_q$, q be the qualities of goods obtained in even units of quantities $\tilde{\delta}_1, \ldots, \tilde{\delta}_q$ respectively.
- $\tilde{x}_{ijkl}$ $(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$, be the quantity of product H_l transported from the origin A_i towards the destination B_j using the means of transportation S_k.
- $\tilde{c}_{ijkl}$ $(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$, be the unit cost of transport of product $\tilde{x}_{ijkl}$.

Mathematically, a four index fuzzy transportation problem can be stated as follows:

$$\begin{cases} \text{Minimize } \tilde{Z} =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl} \\ \text{subject to the constraints:} \\ \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\alpha}_i \text{ for all } \quad i = 1, \ldots, m \\ \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\beta}_j \text{ for all } \quad j = 1, \ldots, n \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\gamma}_k \text{ for all } \quad k = 1, \ldots, p \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \tilde{x}_{ijkl} =_{\Re} \tilde{\delta}_l \text{ for all } \quad l = 1, \ldots, q \\ \tilde{x}_{ijkl} \geq_{\Re} 0, \text{ for all } \quad i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q, \end{cases} \tag{10}$$

where for all (i, j, k, l), we have $\tilde{\alpha}_i >_{\Re} 0$, $\tilde{\beta}_j >_{\Re} 0$, $\tilde{\gamma}_k >_{\Re} 0$, $\tilde{\delta}_l >_{\Re} 0$, and $\tilde{c}_{ijkl} \geq_{\Re} 0$.

We can also write the four index fully fuzzy transportation problem as the following linear program:

$$\begin{cases} \min \tilde{Z} =_{\Re} \tilde{c}^T \otimes \tilde{x} \\ \text{s.c} \\ A\tilde{x} =_{\Re} \tilde{b} \\ \Re(\tilde{x}) \geq 0, \end{cases} \tag{11}$$

where:

- $\tilde{x} = (\tilde{x}_{1111}, \ldots, \tilde{x}_{mnpq})^T \in F(\mathbb{R})^N$.
- $\tilde{c} = (\tilde{c}_{1111}, \ldots, \tilde{c}_{mnpq})^T \in F(\mathbb{R})^N$.
- $\tilde{b} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_n, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_p, \tilde{\delta}_1, \ldots, \tilde{\delta}_q) \in F(\mathbb{R})^M$.
- A is the $M \times N$ matrix with coefficients in $\mathbb{R}$.
- $M = m + n + p + q$ and $N = mnpq$.*

Let $E = \{(i, j, k, l);\ i = 1 : m, j = 1 : n, k = 1 : p \text{ and } l = 1, q\}$ associate for each $(i, j, k, l) \in E$ a vector $P_{ijkl} \in \mathbb{R}^M$. Only four components of the P_{ijkl} vector are non-zero; they are located in the lines $i, m + j, m + n + k$, and $m + n + p + l$ and have 1 as a common value. We define the matrix A as the matrix of vectors P_{ijkl}. Of note, the matrix A is of $rank\ M - 3$.

The problem is to determine $\tilde{x}_{ijkl}$ so that the total cost of transport is minimal.

By generalizing the feasible condition in [27], we obtain the following theorem:

Theorem 1. *The four index fully fuzzy transportation problem has a feasible solution if and only if*

$$\sum_{i=1}^{m} \tilde{\alpha}_i =_{\Re} \sum_{j=1}^{n} \tilde{\beta}_j =_{\Re} \sum_{k=1}^{p} \tilde{\gamma}_k =_{\Re} \sum_{l=1}^{q} \tilde{\delta}_l. \tag{12}$$

4. Resolution

To obtain the fuzzy optimal solution for a four index fully fuzzy transportation problem, we go through two phases:

1. Determining an initial basic feasible solution.
2. Improving a basic feasible solution.

4.1. Phase 1

There are three popular methods used to determine an initial feasible solution to the classical transportation problem. They are the least cost, Russell's approximation, and Vogel's approximation methods. We are going to adapt these methods to the four index transportation problem in a fuzzy context. We denote, FLC4, FRAM4, and FVAM4 the adapted algorithms, respectively.

Now, we introduce the description of each method.

Let E_b be the set of interesting cells. At the beginning of each algorithm, $E_b = \emptyset$.

4.1.1. FLC4 algorithm

The FLC4 algorithm is based on the least cost cell method; the principle of this method is to determine in each step the quantity transported with the minimum cost.
Step 1:

1. For each unsaturated quadruplet (i, j, k, l), choose $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$, where $\tilde{c}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min \tilde{c}_{ijkl}$.
2. Take $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min\left(\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}, \tilde{\delta}_{\bar{l}}\right)$ and add $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ to E_b.
3. Update $\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}$ and $\tilde{\delta}_{\bar{l}}$ as follows:
   (a) $\tilde{\alpha}_{\bar{i}} = \tilde{\alpha}_{\bar{i}} \ominus \tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   if $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \tilde{\alpha}_{\bar{i}}$ then let $\tilde{x}_{\bar{i}jkl}$ be equal to fuzzy zero and saturate $\tilde{c}_{\bar{i}jkl}$, $\forall (j, k, l) \neq (\bar{j}, \bar{k}, \bar{l})$.
   (b) $\tilde{\beta}_{\bar{j}} = \tilde{\beta}_{\bar{j}} \ominus \tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   if $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \tilde{\beta}_{\bar{j}}$ then let $\tilde{x}_{i\bar{j}kl}$ be equal to fuzzy zero and saturate $\tilde{c}_{i\bar{j}kl}$, $\forall (i, k, l) \neq (\bar{i}, \bar{k}, \bar{l})$.
   (c) $\tilde{\gamma}_{\bar{k}} = \tilde{\gamma}_{\bar{k}} \ominus \tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   if $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \tilde{\gamma}_{\bar{k}}$ then let $\tilde{x}_{ij\bar{k}l}$ be equal to fuzzy zero and saturate $\tilde{c}_{ij\bar{k}l}$, $\forall (i, j, l) \neq (\bar{i}, \bar{j}, \bar{l})$.
   (d) $\tilde{\delta}_{\bar{l}} = \tilde{\delta}_{\bar{l}} \ominus \tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   if $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \tilde{\delta}_{\bar{l}}$ then let $\tilde{x}_{ijk\bar{l}}$ be equal to fuzzy zero and saturate $\tilde{c}_{ijk\bar{l}}$, $\forall (i, j, k) \neq (\bar{i}, \bar{j}, \bar{k})$.

Step 2:

- Repeat from 1 to 3 until all $\tilde{x}_{ijkl}$ variables are determined.

4.1.2. FRAM4 algorithm

The FRAM4 algorithm is based on Russell's approximation method [20]. The idea of this method is to determine in each step the minimum reduced cost matrix. Then, the transported quantity is chosen as the minimum of this matrix.
Step 1:

1. For each i unsaturated, determine $\tilde{c}^i = \max\limits_{jkl} \tilde{c}_{ijkl}$.
2. For each j unsaturated, determine $\tilde{c}^j = \max\limits_{ikl} \tilde{c}_{ijkl}$.
3. For each k unsaturated, determine $\tilde{c}^k = \max\limits_{ijl} \tilde{c}_{ijkl}$.
4. For each l unsaturated, determine $\tilde{c}^l = \max\limits_{ijk} \tilde{c}_{ijkl}$.
5. Calculate the reduced cost matrix $c^*_{ijkl} = \tilde{c}_{ijkl} \ominus (\tilde{c}^i \oplus \tilde{c}^j \oplus \tilde{c}^k \oplus \tilde{c}^l)$.
6. Choose the cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ with the smallest fuzzy reduced cost c^*_{ijkl}; if there is more than one, choose the one with the smallest fuzzy cost $\tilde{c}_{ijkl}$; if there is equality again, choose the one whose $\min(\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\gamma}_k, \tilde{\delta}_l)$ is the largest.
7. Take $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min\left(\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}, \tilde{\delta}_{\bar{l}}\right)$ and add $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ to E_b.
8. Update $\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}$ and $\tilde{\delta}_{\bar{l}}$ as in step 1 (3) in the FLC4 algorithm.

Step 2:

- Repeat from 1 to 8 until all $\tilde{x}_{ijkl}$ variables are determined.

4.1.3. FVAM4 algorithm

The FVAM4 algorithm is based on Vogel's approximation method. This method relies on the minimization of a system of penalties. A penalty of dimension i, j, k or l is the fuzzy difference between the smallest and next smallest fuzzy cost.

Step 1:

1. For each i unsaturated, determine the penalty $p_i^1 = \min_2^i \ominus \min_1^i$.
2. For each j unsaturated, determine the penalty $p_j^2 = \min_2^j \ominus \min_1^j$.
3. For each k unsaturated, determine the penalty $p_k^3 = \min_2^k \ominus \min_1^k$.
4. For each l unsaturated, determine the penalty $p_l^4 = \min_2^l \ominus \min_1^l$.
5. Identify the dimension corresponding to the highest penalty.
6. In the selected dimension found in previous step, identify the cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ with the smallest fuzzy cost.
7. Take $\tilde{x}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min(\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}, \tilde{\delta}_{\bar{l}})$ and add $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ to E_b.
8. Update $\tilde{\alpha}_{\bar{i}}, \tilde{\beta}_{\bar{j}}, \tilde{\gamma}_{\bar{k}}$ and $\tilde{\delta}_{\bar{l}}$ as in step 1 (3) in the FLC4 algorithm.

Step 2:

- Repeat from 1 to 8 until all $\tilde{x}_{ijkl}$ variables are determined.

4.2. Treatment of degeneracy

At the end of phase 1, we obtain the initial feasible solution, which can be degenerate or not degenerate.

Let $I = \{(i, j, k, l);\ \Re(\tilde{x}_{ijkl}) > 0\}$, and A_x is the matrix of vectors $P_{ijkl} \forall (i, j, k, l) \in E_b$.

4.2.1. Test of degeneracy

- **if** $rank(A_x) = rank(A)$ **then**
  the obtained solution is not degenerate.
- **elseif** $rank(A_x) < rank(A)$ **then**
  the obtained solution is degenerate.

4.2.2. Treatment of degeneracy

Let N_b be the number of elements of E_b that are in the solution, let E_h be the complement of E_b in the ensemble E (i.e., $E_h = E_b^c$), and let $s = rank(A) - N_b$.

- **if** $N_b = rank(A)$ (i.e., $s = 0$), **then**
  the base is complete and $I^{(0)} = E_b$
- **if** $N_b < rank(A)$ (i.e., $s > 0$), **then**
  the solution is degenerate; we define the ensemble E_s with s elements from E_h that are chosen randomly until $E_b \cup E_s$ is linearly independent and take $I^{(0)} = E_b \cup E_s$. It is a modification of the method in [28].

4.3. Phase 2

To test the optimality or to improve the basic feasible solution, we will adapt the phase 2 of AL_{PT4C} [27].

1. Initialization:

   Let $I^{(r)}$ be the set of interesting quadruplet (i, j, k, l) in iteration r. First, take **r=0**; $I^{(0)}$ was previously defined.

2. For all $(i, j, k, l) \in I^{(r)}$, solve the linear system

$$A_x Y =_{\Re} \tilde{c}_{ijkl} \quad \text{where} \quad Y = [\tilde{u}_i^{(r)}, \tilde{v}_j^{(r)}, \tilde{w}_k^{(r)}, \tilde{t}_l^{(r)}]; i = 1 : m, j = 1 : n, k = 1 : p, l = 1 : q.$$

3. For all $(i, j, k, l) \notin I^{(r)}$ determine

$$\tilde{\Delta}_{ijkl}^{(r)} = \tilde{c}_{ijkl} \ominus (\tilde{u}_i^{(r)} \oplus \tilde{v}_j^{(r)} \oplus \tilde{w}_k^{(r)} \oplus \tilde{t}_l^{(r)})$$

.

4.
   - **If** $\forall(i, j, k, l) \notin I^{(r)}$, we have $\Re\left(\tilde{\Delta}_{ijkl}^{(r)}\right) \geq 0$ **then** the solution $\tilde{x}^{(r)}$ is **optimal**. **Stop**.
   - **Else** use

$$\tilde{\Delta}_{i_0 j_0 k_0 l_0}^{(r)} = \min\{\tilde{\Delta}_{ijkl}^{(r)};\ \ \Re\left(\tilde{\Delta}_{ijkl}^{(r)}\right) < 0\}.$$

   (a) For all $(i, j, k, l) \in I^{(r)}$, determine a cycle $\mu^{(r)}$ by solving the system

$$\sum \lambda_{ijkl}^{(r)} A_x = -P_{i_0 j_0 k_0 l_0}$$

   (b) Determine $\theta^{(r)}$ where $\theta^{(r)} = \min\ (\frac{\tilde{x}_{ijkl}^{(r)}}{-\lambda_{ijkl}^{(r)}}) = \ \theta_{i_s j_s k_s l_s}^{(r)}$ with $\lambda_{ijkl}^{(r)} < 0$

   (c) Determine a new set of basic solution $\tilde{x}^{(r+1)}$ and basic cells as follows:

$$\tilde{x}^{(r+1)} = \left\{\tilde{x}_{ijkl}^{(r)} / (i, j, k, l) \notin \mu^{(r)}\right\} \cup \left\{\tilde{x}_{ijkl}^{(r)} \oplus \lambda_{ijkl}^{(r)} \theta^{(r)} / (i, j, k, l) \in \mu^{(r)}\right\}.$$

$$I^{(r+1)} = \left\{I^{(r)} \cup \{(i_0, j_0, k_0, l_0)\}\right\} \setminus \{(i_s, j_s, k_s, l_s\}$$

   (d) Repeat steps **1)**. . . **4)**.

5. Numerical implementation and comparative study

In this section, we will provide a numerical experiment to test the effectiveness of our adapted algorithms.

Of note, for simplicity, the ranking function of a fuzzy number (a, b, c) is $\Re$ instead of $\Re(a, b, c)$.

5.1. Example

In the following example, $\tilde{x}_B^{(r)}$ and $\tilde{x}_H^{(r)}$ are the set of basic variables and non-basic variables at iteration r, respectively.

The transportation problem is in the form $FFTP_4$ with $(m = n = p = q = 2)$, whose quantities $\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\gamma}_k, \tilde{\delta}_l$, and $\tilde{c}_{ijkl}$ are given by the following tables.

$\tilde{c}_{1111}$	$\tilde{c}_{1112}$	$\tilde{c}_{1121}$	$\tilde{c}_{1122}$	$\tilde{c}_{1211}$	$\tilde{c}_{1212}$	$\tilde{c}_{1221}$	$\tilde{c}_{1222}$
$(4,5,6)$ $\Re=5$	$(0,2,7)$ $\Re=2.75$	$(1,3,5)$ $\Re=3$	$(5,6,9)$ $\Re=6.5$	$(3,5,6)$ $\Re=4.75$	$(4,6,9)$ $\Re=6.25$	$(6,7,9)$ $\Re=7.25$	$(2,4,5)$ $\Re=3.75$
$\tilde{c}_{2111}$	$\tilde{c}_{2112}$	$\tilde{c}_{2121}$	$\tilde{c}_{2122}$	$\tilde{c}_{2211}$	$\tilde{c}_{2212}$	$\tilde{c}_{2221}$	$\tilde{c}_{2222}$
$(5,6,8)$ $\Re=6.25$	$(6,8,10)$ $\Re=8$	$(2,3,7)$ $\Re=3.75$	$(6,8,12)$ $\Re=8.5$	$(7,9,11)$ $\Re=9$	$(3,9,7)$ $\Re=7$	$(3,4,5)$ $\Re=4$	$(4,6,10)$ $\Re=6.5$

Table 1: *Matrix of costs*

$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	$\tilde{\beta}_1$	$\tilde{\beta}_2$	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\delta}_1$	$\tilde{\delta}_2$
$(3,7,7)$ $\Re=6$	$(1,2,7)$ $\Re=3$	$(3,4,8)$ $\Re=4.75$	$(1,5,6)$ $\Re=4.25$	$(2,2,3)$ $\Re=2.25$	$(2,7,11)$ $\Re=6.75$	$(0,4,6)$ $\Re=3.5$	$(4,5,8)$ $\Re=5.5$

Table 2: *Table of $\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\gamma}_k$ and $\tilde{\delta}_l$ quantities*

This $FFTP_4$ provides a feasible solution because:

$$\sum_{i=1}^{2} \tilde{\alpha}_i =_{\Re} \sum_{j=1}^{2} \tilde{\beta}_j =_{\Re} \sum_{l=1}^{2} \gamma_k =_{\Re} \sum_{l=1}^{2} \tilde{\delta}_l = (4,9,14)$$

The size of this problem is $(2,2,2,2)$; thus, here, $M = 8$ and $N = 16$.

5.1.1. Application of the FLC4 algorithm

Step 1

- Take $E_b = \emptyset$,
- $\min \tilde{c}_{ijkjl} = \tilde{c}_{1112}$,
- Determine $\tilde{x}_{1112}$:

$$\begin{aligned}\tilde{x}_{1112} &= \min\left(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_2\right)\\ &= \min\left((3,7,7)_{\Re=6}, (3,4,8)_{\Re=4.75}, (2,2,3)_{\Re=2.25}, (4,5,8)_{\Re=5.5}\right)\\ &= (2,2,3)_{\Re=2.25}.\end{aligned}$$

- Add $(1,1,1,2)$ to E_b,
- Update $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1$ and $\tilde{\delta}_2$ as:

$$\tilde{\alpha}_1 = (0,5,5),$$
$$\tilde{\beta}_1 = (0,2,6),$$
$$\tilde{\gamma}_1 = (-1,0,1),$$
$$\tilde{\delta}_2 = (1,3,6).$$

- For all $(i,j,l) \neq (1,1,2)$, let $\tilde{x}_{ij1l}$ be equal to fuzzy zero and saturate $\tilde{c}_{ij1l}$.

Then, repeat until all $\tilde{x}_{ijkl}$ variables are determined.

The initial basic feasible solution given by FLC4 is: $\tilde{x}^{(0)} = \tilde{x}_B^{(0)} \cup \tilde{x}_H^{(0)}$ where

$$\tilde{x}_B^{(0)} = \left\{\tilde{x}_{1112}^{(0)} = (2,2,3), \tilde{x}_{1121}^{(0)} = (0,2,6), \tilde{x}_{1222}^{(0)} = (-6,3,5), \tilde{x}_{2221}^{(0)} = (-6,2,6), \tilde{x}_{2222} = (-4,0,12)\right\}.$$

The value of the objective associated with $\tilde{x}^{(0)}$ is

$$\tilde{Z}^{(0)} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \tilde{c}_{ijkl} \otimes \tilde{x}^{(0)}_{ijkl} = (16, 33, 59.5).$$

Test of degeneracy
The number of elements of $\tilde{x}^{(0)}_B$ is equal to $5 = M - 3$; thus, the solution is not degenerate.
Phase 2
The test of optimality in phase 2 shows that this solution is not optimal. Thus, we can improve it. $\tilde{x}^{(1)} = \tilde{x}^{(1)}_B \cup \tilde{x}^{(1)}_H$ where
$\tilde{x}^{(1)}_B = \left\{\tilde{x}^{(1)}_{1112} = (2,2,3), \tilde{x}^{(1)}_{1121} = (-12,2,10), \tilde{x}^{(1)}_{1222} = (-10,3,17), \tilde{x}^{(1)}_{2121} = (-4,0,12), \tilde{x}^{(1)}_{2221} = (-6,2,6)\right\}.$
The test of optimality shows that $\tilde{x}^{(1)}$ is optimal. The value of the objective associated with $\tilde{x}^{(1)}$ is:

$$\tilde{Z}^{(1)} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \tilde{c}_{ijkl} \otimes \tilde{x}^{(1)}_{ijkl} = (14, 29, 53.5).$$

5.1.2. Application of the FRAM4 algorithm

- Take $E_b = \emptyset$,
- Calculate the reduced costs:

$\tilde{c}^{(i=1)}$	$\tilde{c}^{(i=2)}$	$\tilde{c}^{(j=1)}$	$\tilde{c}^{(j=2)}$	$\tilde{c}^{(k=1)}$	$\tilde{c}^{(k=2)}$	$\tilde{c}^{(l=1)}$	$\tilde{c}^{(l=2)}$
$(6,7,9)$	$(7,9,11)$	$(6,8,12)$	$(7,9,11)$	$(7,9,11)$	$(6,8,12)$	$(7,9,11)$	$(6,8,12)$

Table 3: *Reduced costs in the first iteration*

- Calculate the reduced costs matrix:

c^*_{1111}	c^*_{1112}	c^*_{1121}	c^*_{1122}
$(-29,-28,-20)$	$(-44,-30,-18)$	$(-43,-29,-20)$	$(-40,-25,-15)$
c^*_{1211}	c^*_{1212}	c^*_{1221}	c^*_{1222}
$(-39,-29,-21)$	$(-39,-27,-17)$	$(-37,-26,-17)$	$(-42,-28,-20)$
c^*_{2111}	c^*_{2112}	c^*_{2121}	c^*_{2122}
$(-40,-29,-19)$	$(-40,-26,-16)$	$(-44,-31,-19)$	$(-41,-25,-13)$
c^*_{2211}	c^*_{2212}	c^*_{2221}	c^*_{2222}
$(-37,-27,-17)$	$(-42,-26,-20)$	$(-42,-31,-22)$	$(-44,-28,-16)$

Table 4: *Reduced costs matrix in the first iteration*

- $\min_{i,j,k,l} c^*_{ijkl} = c^*_{2221}$,
- Determine $\tilde{x}_{2221}$:

$$\begin{aligned}\tilde{x}_{2221} &= \min\left(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2, \tilde{\delta}_1\right)\\ &= \min\left((1,2,7)_{\Re=3}, (1,5,6)_{\Re=4.25}, (2,7,11)_{\Re=6.75}, (0,4,6)_{\Re=3.5}\right)\\ &= (1,2,7)_{\Re=3}.\end{aligned}$$

- Add $(2, 2, 2, 1)$ to E_b,
- Update $\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2$ and $\tilde{\delta}_1$ as:

$$\tilde{\alpha}_2 = (-6, 0, 6),$$
$$\tilde{\beta}_2 = (-6, 3, 5),$$
$$\tilde{\gamma}_2 = (-5, 5, 10),$$
$$\tilde{\delta}_1 = (-7, 2, 5).$$

- For all $(i, j, l) \neq (2, 2, 1)$, let $\tilde{x}_{2jkl}$ be equal to fuzzy zero and saturate $\tilde{c}_{2jkl}$.

Then, repeat until all $\tilde{x}_{ijkl}$ variables are determined.

The initial basic feasible solution given by the FRAM4 method is $\tilde{x}^{(0)} = \tilde{x}_B^{(0)} \cup \tilde{x}_H^{(0)}$ where

$$\tilde{x}_B^{(0)} = \left\{\tilde{x}_{2221}^{(0)} = (1, 2, 7), \tilde{x}_{1121}^{(0)} = (-7, 2, 5), \tilde{x}_{1112}^{(0)} = (2, 2, 3), \tilde{x}_{1222}^{(0)} = (-6, 3, 5), \tilde{x}_{1122}^{(0)} = (-10, 0, 18)\right\}.$$

The value of the objective associated with $\tilde{x}^{(0)}$ is

$$\tilde{Z}^{(0)} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}^{(0)} = (14, 29, 53.5).$$

Test of degeneracy

The number of elements of $\tilde{x}_B^{(0)}$ is equal to $5 = M - 3$; thus, the solution is not degenerate.

Phase 2

The test of optimality in phase 2 shows that the initial solution obtained from the FRAM4 algorithm is optimal and does not need improvement.

5.1.3. Application of the FVAM4 algorithm

Step 1

- Take $E_b = \emptyset$,
- Calculate the penalties:

p_1^1	p_2^1	p_1^2	p_2^2	p_1^3	p_2^3	p_1^4	p_2^4
$(-6, 1, 5)$	$(-4, 1, 3)$	$(-6, 1, 5)$	$(-2, 0, 3)$	**(-4,3,6)**	$(-3, 1, 4)$	$(-3, 0, 6)$	$(-5, 2, 5)$

Table 5: *Penalties in the first iteration*

- $\max(p_i^1, p_j^2, p_k^3, p_l^4) = p_1^3$,
- $\min \tilde{c}_{ij1l} = \tilde{c}_{1112}$,
- Determine $\tilde{x}_{1112}$:

$$\begin{aligned}\tilde{x}_{1112} &= \min\left(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_2\right) \\ &= \min\left((3, 7, 7)_{\Re=6}, (3, 4, 8)_{\Re=4.75}, (2, 2, 3)_{\Re=2.25}, (4, 5, 8)_{\Re=5.5}\right) \\ &= (2, 2, 3)_{\Re=2.25}.\end{aligned}$$

- Add $(1, 1, 1, 2)$ to E_b,

- Update $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1$ and $\tilde{\delta}_2$ as:

$$\tilde{\alpha}_1 = (0, 5, 5),$$
$$\tilde{\beta}_1 = (0, 2, 6),$$
$$\tilde{\gamma}_1 = (-1, 0, 1),$$
$$\tilde{\delta}_2 = (1, 3, 6).$$

- For all $(i, j, l) \neq (1, 1, 2)$, let $\tilde{x}_{ij1l}$ be equal to fuzzy zero and saturate $\tilde{c}_{ij1l}$.

Then repeat until all $\tilde{x}_{ijkl}$ variables are determined.

The initial basic feasible solution given by the FVAM4 method is: $\tilde{x}^{(0)} = \tilde{x}_B^{(0)} \cup \tilde{x}_H^{(0)}$, where

$$\tilde{x}_B^{(0)} = \left\{ \tilde{x}_{1112}^{(0)} = (2, 2, 3), \tilde{x}_{1222}^{(0)} = (1, 3, 6), \tilde{x}_{1121}^{(0)} = (-6, 2, 4), \tilde{x}_{2221}^{(0)} = (-5, 2, 5), \tilde{x}_{2121}^{(0)} = (-4, 0, 12) \right\}.$$

The value of the objective associated with $\tilde{x}^{(0)}$ is

$$\tilde{Z}^{(0)} =_{\Re} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} \tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}^{(0)} = (14, 29, 53.5)$$

Test of degeneracy

The number of elements of $\tilde{x}_B^{(0)}$ is equal to $5 = M - 3$; thus, the solution is not degenerate.

Phase 2

The test of optimality in phase 2 shows that $\tilde{x}^{(0)}$ is optimal and does not need improvement.

5.2. Comparative study

The following tables show a comparative study between the three adapted algorithms FLC4, FRAM4, and FVAM4 that are applied to solve the four index fully fuzzy transportation problem. We will treat 13 different problems with different dimensions $M \times N$. For each problem, a set of data and cost matrixes are randomly chosen for the fair comparison of these algorithms.

Discussion

- These tables show that the proposed algorithms can effectively solve $FFTP_4$ with a large range of dimensions from 8×16 to more than 113×630000.
- The results show the robustness of our method in phase 2 when finding the optimal solution, from the initial one, even in the case of degeneracy.
- The initial basic feasible solution obtained by FLC4 is far from being the optimal one for most $FFTP_4$; its main advantage is that it is quick and easy, and its simulation code is very simple.

Dimension ($M \times N$)			12×81	16×265	20×625	24×1296	32×4096	40×1000	64×65536
phase 1	$\tilde{Z}^{(0)}$	FLC4	(-2998.8 2093.8 7131.8)	(-21125 1653.3 24298)	(-54986 1052.3 57223)	(-75878 2136.3 79614)	(-62829 1166.5 65238)	(-33481 348.5 34377)	(-77002 1750 80003)
		FRAM4	(-1948.8 2043.8 6307)	(-23479 1622.5 26517)	(-70631 931 73088)	(-45375 1895.8 49255)	(-40825 1021.3 43588)	(-61920 314.75 63186)	(-1.4375e5 1432 1.4748e5)
		FVAM4	(-2836.3 2103.3 6971.3)	(-29170 1440.5 32234)	(-25078 1094.8 26875)	(-43860 1865.8 47852)	(-60253 1192.3 62500)	(-84944 360 85863)	(-1.1649e5 1417.3 1.199e5)
	$\Re(\tilde{Z}^{(0)})$	FLC4	2080.1	1619.9	1085.6	2002.3	1185.5	398.25	1625.3
		FRAM4	2111.4	1570.8	1079.8	1917.7	1201.4	474	1649.1
		FVAM4	2085.4	1486.4	996.69	1931	1158	409.69	1561
	time(ms)	FLC4	1.5356	2.2069	4.0085	12.481	13.846	15.576	17.398
		FRAM4	3.0169	6.7252	9.512	31.602	50.462	96.994	701.71
		FVAM4	3.6408	6.794	7.9264	30.963	30.922	45.034	168.96
phase 2	$\tilde{Z}^*$	FLC4	(-2818 2015.8 6833.8)	(-12709 1343.3 15325)	(-27951 864.25 29806)	(-52205 1455.3 54931)	(-28621 847.66 30335)	(-18536 284.85 19198)	(-43864 682.97 45460)
		FRAM4	(-1900.8 2015.7 5916.5)	(-10040 1347.3 12649)	(-32677 874.75 34511)	(-28878 1455.3 31605)	(-19155 847.66 20869)	(-41649 295.58 42290)	(-72237 704.61 73789)
		FVAM4	(-2775.2 2015.8 6791)	(-17652 1345.6 20264)	(-17330 868.75 19176)	(-29654 1455.3 32381)	(-37737 848.3 39450)	(-31037 290.22 31689)	(-79578 684.73 81170)
	$\Re(\tilde{Z}^*)$	FLC4	2011.8	1325.9	895.87	1409.3	852.24	308.01	740.36
		FRAM4	2011.8	1325.9	895.87	1409.3	852.24	308.0	740.36
		FVAM4	2011.8	1325.9	895.87	1409.3	852.24	308.01	740.36
	iteration	FLC4	4	12	14	50	52	69	345
		FRAM4	3	14	16	48	70	68	418
		FVAM4	2	8	12	48	57	68	354
	time(ms)	FLC4	8.8915	50.207	145.4	1507.8	8845.3	56340	1.0578e5
		FRAM4	7.1789	58.461	162.94	1347.3	11605	58289	1.2892e5
		FVAM4	5.0874	34.99	125.01	1318.6	9576.5	56062	1.0562e5
Total time (ms)		FLC4	10.427	52.414	149.41	1520.3	8859.1	56356	1.058e5
		FRAM4	10.196	65.186	172.45	1378.9	11655	58386	1.2962e5
		FVAM4	8.7282	41.784	132.94	1349.6	9607.4	56107	1.0579e5

Table 6: FLC4, FRAM4, *and* FVAM4 *in solving* $FFTP_4$

- The initial basic feasible solution obtained by FVAM4 or FRAM4 methods is very close to the optimal one.
- In most cases, FVAM4 requires less time and a smaller number of iterations compared to FRAM4 and FLC4. Consequently, this method is preferable to use to solve the four index fully fuzzy transportation problems with large size.

Dimension ($M \times N$)			30×3136	35×5760	37×7200	41×10800	80×16000	113×630000
phase 1	$\tilde{Z}^{(0)}$	FLC4	(-1.4195e5 1507.8 1.4447e5)	(-30855 2086.8 34891)	(-57572 1157.3 59759)	(-1.2796e5 1602 1.3161e5)	(-2.6674e5 764.25 2.7078e5)	(-5.4236e5 854 5.4413e5)
		FRAM4	(-15678 1711.3 18334)	(-26876 1920 30676)	(-39550 1127.3 41836)	(-57375 1496.3 60930)	(-6.2504e5 1357 6.2814e5)	(-1.9974e5 920.25 2.0114e5)
		FVAM4	(-30597 1508.5 33267)	(-49972 1777.8 52403)	(-38357 1067.8 40487)	(-47532 1353.8 51708)	(-1.5441e5 1256 1.5731e5)	(-234661 885.25 236372)
	$\Re(\tilde{Z}^{(0)})$	FLC4	1384.1	2052.3	1125.4	1714.4	1393.4	871.31
		FRAM4	1519.4	1910.1	1135.2	1636.8	1454.9	808.81
		FVAM4	1421.9	1496.6	1066.4	1721	1352.8	870.38
	time(ms)	FLC4	13.73	6.0158	15.905	16.345	63.173	211.78
		FRAM4	43.844	62.404	75.093	101.12	2148.7	15467
		FVAM4	30.119	24.991	39.1	72.504	382.51	2137
phase 2	$\tilde{Z}^*$	FLC4	(-76187 1080.6 78056)	(-16392 853.97 17984)	(-46683 800.33 48381)	(-72990 933.75 75036)	(-83399 555.63 84528)	(-2.375e5 417.71 2.3825e5)
		FRAM4	(-10714 1108.6 12528)	(-15701 853.97 17293)	(-25600 795.83 27307)	(-39633 940.25 41666)	(-2.8724e5 530.74 2.8842e5)	(-96678 412.08 97440)
		FVAM4	(-22150 1137.7 23905)	(-29421 871.09 30979)	(-34851 794 36562)	(-47532 1353.8 51708)	(-97204 540.54 98363)	(-1.4737e5 418.57 1.4812e5)
	$\Re(\tilde{Z}^*)$	FLC4	1007.6	825.09	824.65	978.38	560	396.65
		FRAM4	1007.6	825.09	824.65	978.38	560	396.65
		FVAM4	1007.6	825.09	824.65	978.38	560	396.65
	iteration	FLC4	58	86	83	130	565	1225
		FRAM4	68	86	79	123	549	1095
		FVAM4	60	83	88	91	512	1074
	time(ms)	FLC4	1103.7	3452.1	3785	7341	4.3418e5	3.6585e6
		FRAM4	1959.5	2813.9	3334.3	8074.2	4.1115e5	3.1405e6
		FVAM4	1245.2	4161.7	3704.6	5824.4	3.9497e5	3.01e6
Total time (ms)		FLC4	1117.4	3458.1	3800.9	7357.4	4.3424e5	3.6587e6
		FRAM4	2003.4	2876.3	3409.4	8175.3	4.133e5	3.156e6
		FVAM4	1275.3	4186.7	3743.7	5896.9	3.9535e5	3.0121e6

Table 7: FLC4, FRAM4, *and* FVAM4 *in solving* $FFTP_4$

6. Conclusion

In many real-life situations, data from transportation problems are often uncertain. To deal with such problems, this uncertainty can be modeled using fuzzy mathematics. In this study, we proposed three approaches (i.e., FLC4, FRAM4, and FVAM4) to determine the initial basic feasible solution to the four index fully fuzzy transportation. Then, after treating the degeneracy problem, we initiate the second phase to determine the optimal solution. Of note, the arithmetic operations used are based on the notion of ranking function. We performed numerical experiments to test the efficiency and stability of our algorithms. The obtained results are encouraging and show that the method, in general, provides the initial solution close to the optimum. The algorithms FLC4, FRAM4, FVAM4 are independent of the number of indexes, as shown in the numerical examples. Therefor a comparative table can be used to solve fuzzy multi-index problems. The second phase of the algorithm has shown its robustness and efficiency for determining the optimal solution in a considerably shorter period of time.

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