

The set of all the possible compromises of a multi-level multi-objective linear programming problem

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Abstract. In this paper, a new approach is developed to solve multi-level multi-objective linear programming problems (ML-MOLPP), by providing the set of all the compromises without taking into account the hierarchy of the problem. A simple criterion is developed for testing whether a given feasible solution is a possible compromise or not.

We describe a method used to generate the set of all the compromises, based on the P.L. Yu and M. Zeleny's method. Theoretical results, numerical example and flow diagram are reported.

Keywords: Multi-level multi-objective linear programming, decision-makers, non-dominated points, non-dominated facets, non-dominated extreme points.

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1. Introduction

Several decision problems are modeled by mathematical programming problems, whether in physics, industry, biology, economics or other fields; optimization methods are then used to solve these problems. At the beginning, problems were modeled with a single objective function, in which only one decision-maker was involved. Then, problems were discovered in which a single decision-maker seeks to optimize several objectives that are generally in conflict, these are multi-objective programming problems, see [4, 7]. It was therefore necessary to specify a sense of resolution and then develop methods of resolution. Other decision-making problems exist when several decision-makers are involved in the decision-making process; each decision-maker seeks to solve a problem with several objectives. These types of problems are modeled by multi-level multi-objective programming problems, see [6, 8, 1].

Multi-level optimization appears to be a very suitable tool to model decision-making problems where multiple decision-makers interact in a hierarchical structure. Mathematically, in a multi-level programming problem, the higher level variables are considered as parameters in the lower level programming problems and the problems of the higher level are regarded as constraints of the lower level. Multi-level programming problems have several applications in different branches such as supply chain management, network defense, planning, logistics,

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economics, government, autonomous institutions, agriculture, army, management, schools, hospitals, banks, etc . . . Although most research on multi-level programming has focused on cases with only two levels (called two-level programming), see [6]. There are many programming problems that involve more than two levels. Since the pioneering work of Bracken and McGill [2, 3], several researchers have published monographs and literature reviews in which theoretical and methodological aspects of two-level optimization were discussed. Several approaches have been developed to solve the multi-level programming problems. For a good bibliography of these problems and their applications, see [5].

In this paper, we consider a ML-MOLPP where the objective functions and the constraints are linear. We exploit the algorithms developed by P.L. Yu and M. Zeleny in [10] page 450 and 461, to develop a new algorithm to generate the entire set of all the possible compromises.

So, in this paper, we propose an adaptation of the methods given in [10] to generate the entire set of possible compromise solutions. In other words, we aim to provide decision-makers involved in decision-making with a set containing all feasible solutions that are non-dominated solutions for all their multi-objective problems. First, we define the set of all non-dominated feasible solutions (set of all possible compromise solution) for a ML-MOLPP and prove that it can be decomposed into non-dominated facets of the feasible region. So, we develop an algorithm to generate the entire non-dominated feasible solutions set. The algorithm uses the P.L. Yu and M. Zeleny's method for generating non-dominated facets of multi-objective linear programming problem (MOLPP), see [10]. Next, we define a sorting set as the facets (convex subsets) constituting the non-dominated feasible solutions set.

The paper is organized as follows. In the next section, we develop the mathematical formulation of the problem. In Section 3, we present some tools of multi-objective linear programming, that is non-dominated solutions and some necessary results about non-dominated facets of a given feasible region. In Section 4, we present the algorithm to generate the entire set of all possible compromise solutions of a ML-MOLPP. In Section 5, we demonstrate the approach with a numerical example. Finally, a conclusion is given in Section 6.

2. Problem formulation

Consider a P -level linear programming problem ($P \geq 2$) and denote DM_p the decision maker at p^{th} level that has control over the decision variables $\bar{x}^p = x_{p1}, \dots, x_{pn_p} \in \mathbb{R}^{n_p}$, $p = 1, \dots, P$, where $x = (\bar{x}^1, \dots, \bar{x}^P)^t$, $n = n_1 + \dots + n_P$ and $k = k_1 + \dots + k_P$.

We define

$$F_p : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_P} \longmapsto \mathbb{R}^{k_p} \\ x \longmapsto F_p(x) = c_p x, \quad (1)$$

where

$$c_p = \begin{pmatrix} c_{p1} \\ c_{p2} \\ \vdots \\ c_{pk_p} \end{pmatrix} = \begin{pmatrix} c_{p1}^{11} & \dots & c_{p1}^{1n_1} & c_{p1}^{21} & \dots & c_{p1}^{2n_2} & \dots & c_{p1}^{P1} & \dots & c_{p1}^{Pn_P} \\ c_{p2}^{11} & \dots & c_{p2}^{1n_1} & c_{p2}^{21} & \dots & c_{p2}^{2n_2} & \dots & c_{p2}^{P1} & \dots & c_{p2}^{Pn_P} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ c_{pk_p}^{11} & \dots & c_{pk_p}^{1n_1} & c_{pk_p}^{21} & \dots & c_{pk_p}^{2n_2} & \dots & c_{pk_p}^{P1} & \dots & c_{pk_p}^{Pn_P} \end{pmatrix}, \quad p = 1, \dots, P,$$

$$c_{pq}x = c_{pq}^{1j}\bar{x}^1 + c_{pq}^{2j}\bar{x}^2 + \dots + c_{pq}^{Pj}\bar{x}^P, \quad (p = 1, \dots, P, \quad q = 1, \dots, k_p, \quad j = 1, \dots, n_p) \\ = c_{pq}^{11}x_{11} + \dots + c_{pq}^{1n_1}x_{1n_1} + c_{pq}^{21}x_{21} + \dots + c_{pq}^{2n_2}x_{2n_2} + \dots + c_{pq}^{P1}x_{P1} + \dots + c_{pq}^{Pn_P}x_{Pn_P},$$

and

$$c_{pq}^j = c_{pq}^{p1}, c_{pq}^{p2}, \dots, c_{pq}^{pn_p}, \quad p = 1, \dots, P, \quad q = 1, \dots, k_p.$$

The formulation of a P -level multi-objective linear programming problem is given as follows:

Level 1

$$\max_{\bar{x}^1} F_1(x) = \max_{\bar{x}^1} \begin{pmatrix} c_{11}x \\ c_{12}x \\ \vdots \\ c_{1k_1}x \end{pmatrix},$$

such that $\bar{x}^2, \dots, \bar{x}^P$ solve

Level 2

$$\max_{\bar{x}^2} F_2(x) = \max_{\bar{x}^2} \begin{pmatrix} c_{21}x \\ c_{22}x \\ \vdots \\ c_{2k_2}x \end{pmatrix}, \quad (2)$$

\vdots

such that \bar{x}^P solves

Level P

$$\max_{\bar{x}^P} F_P(x) = \max_{\bar{x}^P} \begin{pmatrix} c_{P1}x \\ c_{P2}x \\ \vdots \\ c_{Pk_P}x \end{pmatrix}$$

subject to

$$x \in S = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0, b \in \mathbb{R}^m\},$$

where $S \neq \emptyset$ is the multi-level convex constraints feasible choice set, m is the number of the constraints, k_p is the number of DM_p 's objective functions, k is the number of all the objective functions that constitutes the ML-MOLPP, c_{pq}^{ij} are constants, A is a $(m \times n)$ -matrix and b is a m -vector. We assume that $m < n$ and the feasible region S is a compact polyhedron, which means bounded polyhedron.

Notation 1.

- For any $x \in S$, we use x_i to indicate its i^{th} coordinate, also called its i^{th} component and we define the following sets of indices:

$$I = \{1, 2, \dots, m\}, J = \{1, 2, \dots, n\}, M = \{1, 2, \dots, n + m\}.$$

Let J_B and J_N be two subsets of J verifying

$$J = J_B \cup J_N, J_B \cap J_N = \emptyset, |J_B| = m, |J_N| = n - m,$$

this will allow us to write the partition of x as follows:

$$\begin{aligned} x = x(J) &= (x_j, j \in J), & x &= \begin{pmatrix} x_N \\ x_B \end{pmatrix}, \\ x_N &= x(J_N) = (x_j, j \in J_N), & x_B &= x(J_B) = (x_j, j \in J_B). \end{aligned} \quad (3)$$

The decision variables vectors $x \in S$ will be considered as column vectors, unless otherwise explicitly stated and x^t refers to the transposition of x which is a row vector. The $x > 0$

and $x \geq 0$ ratings indicate that all x components are positive and not negative, respectively. For two vectors x^1 and x^2 , the $x^1 > x^2$ notation means $x^1 - x^2 > 0$. The $x^1 \geq x^2$, $x^1 < x^2$, $x^1 \leq x^2 \dots$ etc, notations should be interpreted accordingly.

- Let $A = (a_{ij})$ be a matrix of size $m_1 \times m_2$, where m_1, m_2 are any integers. We denote by A^t the transposed matrix of A , defined by $A^t = (a_{ji})$.
- For practical calculations, we introduce the following notation of a matrix A :

$$A = (a_1, a_2, \dots, a_j, \dots, a_n) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{pmatrix},$$

where

$$a_j = A(I, j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, j \in J$$

is a column vector of dimension m and is considered as an matrix of order $(m \times 1)$ and

$$A_i = (a_{i1} \ a_{i2} \ \dots \ a_{in}), i \in I$$

is a row vector of dimension n and can be considered as an matrix of order $(1 \times n)$. So, we can write the partition of a matrix A as follows:

$$A = (A_N, A_B), \quad A_N = A(I, J_N), \quad A_B = A(I, J_B). \quad (4)$$

3. Preliminaries

We already know that the feasible region S defined from a linear inequality system is a convex polyhedron, see [10], moreover we have assumed that S is bounded. Then, it is clear that the notion of polyhedron, which is a geometric notion, is naturally attached to the linear optimization models, in particular the multi-objective multi-level linear programming problems that we consider in this work. For more details about the results given here, see [10].

Let's denote by \tilde{A} a $((n + m) \times n)$ -matrix which the first m rows correspond to the rows of the matrix A and the last rows correspond to the n non-negativity constraints ($x_{ij} \geq 0$, $i = 1, \dots, P$, $j = 1, \dots, n_i$), \tilde{b} be a $(n + m)$ -vector with $\tilde{b}_i = b_i$ for $i = 1, \dots, m$ and $\tilde{b}_i = 0$ for $i = m + 1, \dots, m + n$, which means:

$$\tilde{A} = \begin{pmatrix} A \\ -Id_n \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where Id_n is the identity matrix of order n . Consider the linear constraints of the problem (2) (linear system of inequalities), defined by

$$\tilde{A}x \leq \tilde{b}. \quad (5)$$

Then, for all $i \in M$, we have

1. The solution set F^i of the system $\tilde{A}_i x = \tilde{b}_i$ is the affine hyperplane defined by

$$F^i = \left\{ x \in \mathbb{R}^n \mid \tilde{A}_i x = \tilde{b}_i \right\}.$$

2. The solution set S^i of the system $\tilde{A}_i x \leq \tilde{b}_i$ is the affine half-space defined by

$$S^i = \left\{ x \in \mathbb{R}^n \mid \tilde{A}_i x \leq \tilde{b}_i \right\}.$$

3. The feasible region S of the solutions of the inequality system (5) is obtained by the intersection of a finite number of half-spaces S^i of \mathbb{R}^n

$$S = \bigcap_{i=1}^{m+n} S^i.$$

Definition 1.

1. For all $i = 1, \dots, m$, F^i is called boundary hyperplane. Note that a boundary hyperplane does not necessarily meet the polyhedron S , see the Figure 1.
2. The intersection of a boundary hyperplane with the polyhedron S forms a face of S .
3. The intersection of a boundary hyperplane with the polyhedron S containing exactly one point is an extreme point of S .
4. A face is also a polyhedron and a polyhedron is a face of itself.

More generally, for a given $Q \subseteq M$, let \tilde{A}_Q be the matrix derived from \tilde{A} by deleting the rows which are not in Q . Similarly, \tilde{b}_Q is derived. Then, we call a facet of S the set defined by

$$F(Q) = \left\{ x \in S \mid \tilde{A}_Q x = \tilde{b}_Q \right\}.$$

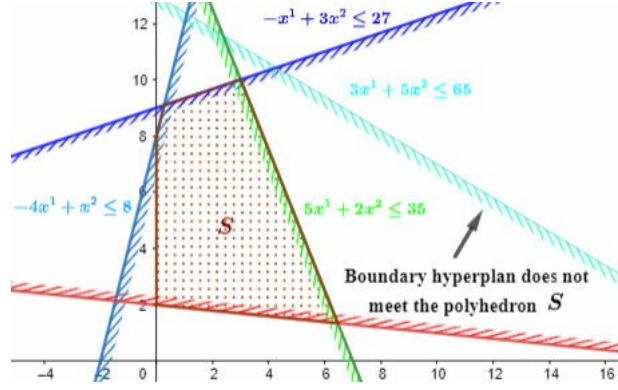
Remark 1. For all $Q \subseteq M$, we have

1. A facet $F(Q)$ of S can be empty.
2. The dimension of a facet $F(Q)$ is the dimension of the affine manifold generated by $F(Q)$.
3. If Q contains only one element, then a facet $F(Q)$ of S is a face. If moreover $F(Q) \neq \emptyset$, then the dimension of $F(Q)$ is equal to $n - 1$.
4. If Q contains more than one element, then the dimension of $F(Q)$ is less or equal to $n - 1$.

Notation 2. Let X be an arbitrary subset of S , then its closure, relative interior (with respect to the relative topology induced in the manifold generated by S) will be denoted by \bar{X} , X' respectively.

Definition 2. Let $F(Q)$ be a facet of a polyhedron S , then

1. The closure $\overline{F(Q)}$ and the relative interior $F(Q)'$ are convex sets.
2. $F(Q)$ is called a proper facet if it is different from the polyhedron S .

Figure 1: Boundary hyperplanes in \mathbb{R}^2

3.1. Dominated and non-dominated feasible solutions

Let $p = 1, \dots, P$ and consider the following multi-objective linear programming problem:

$$\begin{cases} \max & c_p x \\ & Ax \leq b \\ & x \geq 0 \end{cases} \quad (6)$$

where A is a $(m \times n)$ -matrix and b is a m -vector.

Let Z_p be the DM_p 's criteria space defined as follows:

$$Z_p = \{z \in \mathbb{R}^{k_p} : z = c_p x, x \in S\}.$$

Definition 3.

- For $z^1, z^2 \in Z_p$, we say that z^1 dominates z^2 if $z^1 \geq z^2$.
- For the feasible solutions $x^1, x^2 \in S$, we say that x^1 dominates x^2 if $c_p x^1 \geq c_p x^2$.
- A feasible solution $x \in S$ is non-dominated if it is not dominated by any other feasible point of S .

Notation 3.

- N_p denotes the set of all non-dominated feasible solutions and D_p denotes the set of all dominated feasible solutions of the problem (6).
- Since the feasible region S is a compact polyhedron, then it has a finite number of vertices (also called extreme points). We denote the set of all extreme points of S by S^{dex} .
- We denote the set of all non-dominated extreme points (non-dominated feasible solution that belongs to S^{dex}) of the problem (6) by:

$$N_p^{dex} = S^{dex} \cap N_p$$

and N_p^{dex} will denote an arbitrary element of N_p^{dex} .

Lemma 1. Let $x^1, x^2 \in S$ and $x^1 \in D_p$. Then, $[x^1, x^2] \subset D_p$, where

$$[x^1, x^2] = \{\alpha x^1 + (1 - \alpha)x^2 \mid 0 < \alpha \leq 1\}$$

is the line segment bounded by x^1 and x^2 . Specifically, if $x^1 \in S'$ then $S' \subseteq D_p$.

Theorem 1. Let X be an any subset of S , suppose that $x^1 \in X'$ and $x^1 \in N_p$. Then, $\bar{X} \subset N_p$.

Generally, the objectives of a MOLPP are in conflict, which means that for all $p = 1, \dots, P$ we have $S \neq N_p$. Then, from theorem 1 and lemma 1, we get that $S' \subseteq D_p$. This means that the non-dominated feasible points belong to the facets of the feasible region S . This is why the search for non-dominated points of a MOLPP is done on the boundary of the feasible region (S private of S').

Definition 4. Given $Q \subseteq M$, then the set

$$N_p(Q) = N_p \cap F(Q)$$

is called non-dominated facet.

Theorem 2. (Decomposition of N_p into non-dominated facets) Let $Q \subset M$, $l = 0, \dots, m + n$ and

$$\mathcal{F}^l = \{Q \subset M : |Q| = l\}.$$

Then,

$$N_p = \bigcup_{l=0}^{m+n} \bigcap_{Q \in \mathcal{F}^l} N_p(Q). \quad (7)$$

Definition 5. A facet $F(Q)$ is called a full facet, if there is no Q' containing Q such that $F(Q) = F(Q')$. In other words, if $F(Q')$ also represents the facet $F(Q)$, then $Q' \subset Q$.

Theorem 3. Let $F(Q)$ be a full facet such that $N_p^{dex}(Q) := N_p^{dex} \cap F(Q) \neq \emptyset$, then $F(Q) \subset N_p$, only if $N_p^{dex}(Q)$ has at least $n - |Q| + 1$ elements.

After adding the slacks variables to the linear constraints ($Ax \geq b$), we get $Bx = b$, where $B = (A, Id_m)$ and Id_m is the identity matrix of size $m \times m$. To simplify the presentation, we will not change the notation of the decision variable vector x and the matrix c_p . The use of the matrix B means that x is a $(n + m)$ -vector and $c_p = (c_p, 0_{k_p \times m})$, where $0_{k_p \times m}$ is a matrix of size $(k_p \times m)$ with components equal to zero.

Definition 6. Let $J_B \subset M$ such that $|J_B| = m$, $J_N = M - J_B$, B_{J_B} be the non-singular sub-matrix of B (called basis) of size $m \times m$ and B_{J_N} be the remaining sub-matrix. We define

$$x_{J_B} = \left((B_{J_B}^{-1}b)_j : j \in J_B \right) \text{ and } x_{J_N} = (x_j = 0 : j \in J_N).$$

Then, the point $x^0 = (x_{J_B}, x_{J_N})$ (renumber the indexes if necessary) is called basic feasible solution. Then, we use J_B to represent B_{J_B} .

Theorem 4. Let $x^0 = (x_{J_B}, x_{J_N})$ be a basic feasible solution with basis J_B and consider the linear problem

$$w_p = \max \sum_{i=1}^{k_p} v_{pi} \quad (8)$$

subject to

$$\tilde{S}_p = \{(x, v^p) : x \in S, c_p x - v_p \geq c_p x^0, v_p \geq 0\},$$

where

$$v_p = (v_{p1}, \dots, v_{pk_p}).$$

Then,

- x^0 is a non-dominated point, if and only if, $w_p = 0$.
- x^0 is a dominated point, if and only if, $w_p > 0$.

4. Results

Multi-level multi-objective linear programming problems are characterized by the presence of a hierarchy. That is to say, the solution chosen by the decision maker of the lower level depends on the one chosen by the higher level, then the procedures built to obtain a compromise solution of the ML-MOLPP must take this fact into account. So, the hierarchy takes its meaning when we try to determine one and only one compromise, in other words, it is a way of favoring a compromise among other existing ones. On the other hand, sometimes the compromise achieved is not satisfactory and the procedure has to be restarted to find another one, which is sometimes annoying, hence the interest of developing an algorithm to generate all the possible compromise solution. So, to get all possible compromise solutions, we don't need to consider any hierarchy.

In 1975, P.L. Yu and M. Zeleny presented two methods for locating the set of all non-dominated solutions of a MOLPP, the first allows to generate the set of all non-dominated extreme points N_p^{dex} see [9] and the second generates the entire set of all non-dominated solutions N_p through the set of all non-dominated extreme points, see [10].

In a ML-MOLPP each DM_p looks for a non-dominated feasible solution (non-dominated point) in the set of non-dominated solution N_p , so as to maximize its objective function F_p only over the decision variable \bar{x}^p that are over his controls. So, any potential compromise solution \hat{x}^c of a ML-MOLPP is necessarily included in the intersection of the P -sets of non-dominated solutions of the P -MOLPPs which constitutes the P -levels. Indeed, if \hat{x}^c does not belong to the intersection of N_p for $p = 1, \dots, P$, then the compromise \hat{x}^c becomes a dominated solution for at least one decision maker which would contradict the fact that \hat{x}^c is a compromise solution of ML-MOLPP. So, we can naturally define the set of all the compromise solutions (non-dominated feasible solutions) of a ML-MOLPP as follows:

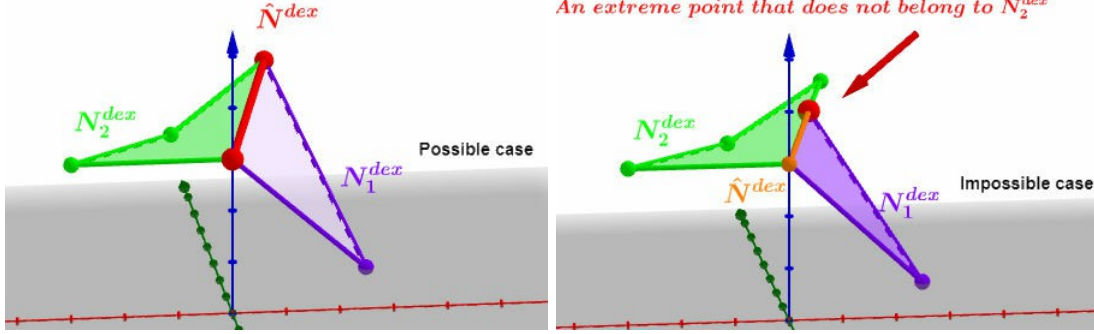
$$\hat{N} = \bigcap_{p=1}^P N_p. \quad (9)$$

Similarly, we define the set of all non-dominated extreme points of a ML-MOLPP as the set of common non-dominated extreme points between all the sets N_p^{dex} , that is:

$$\hat{N}^{dex} = \bigcap_{p=1}^P N_p^{dex}. \quad (10)$$

The definition of the set \hat{N}^{dex} is consistent and does not miss any other non-dominated extreme point. Indeed, since for all $p = 1, \dots, P$, the set of non-dominated points N_p is a polyhedron not necessarily convex obtained as an union of non-dominated facets of S , see [10]. Then, for any two levels p_1, p_2 , the intersection $N_{p_1} \cap N_{p_2}$ can never contain any new non-dominated extreme point that does not belong to both N_{p_1} and N_{p_2} . Which means, all the extreme points of the polyhedron $N_{p_1} \cap N_{p_2}$ are extreme points of both N_{p_1} and N_{p_2} . This is true because N_{p_1} and N_{p_2} are polyhedrons contained in the boundary of another polyhedron S , this is not true in general, see Figures 2.

Remark 2. Since for $p = 1, \dots, P$, N_p is not necessarily a convex set, \hat{N} is not convex in general. If $\hat{N} = \emptyset$, we say that the problem (2) is unsolvable.

Figure 2: Illustrate the set \hat{N}^{dex} for $P = 2$ and $S \subset \mathbb{R}^3$.

4.1. Non-dominated facets of ML-MOLPP

Let assume that $\hat{N} \neq \emptyset$ and define the set of all subsets Q of M that corresponds to non-empty facets $F(Q)$ of the feasible region S , that are contained in \hat{N} , as follows:

$$\hat{\mathcal{N}} = \left\{ Q \subset M \mid F(Q) \neq \emptyset \text{ and } F(Q) \subseteq \hat{N} \right\}.$$

Notation 4. The facet corresponding to a subset Q of $\hat{\mathcal{N}}$ is called non-dominated facet for ML-MOLPP and denoted by

$$\hat{N}(Q) = \bigcap_{p=1}^P N_p(Q).$$

It is clear that for two subsets Q_1 et Q_2 in $\hat{\mathcal{N}}$ verifying $Q_1 \subset Q_2$, we have $F(Q_2) \subseteq F(Q_1)$. In order to eliminate all the facets $F(Q_2)$ that are contained in bigger one $F(Q_1)$, we define a new set $\tilde{\mathcal{N}}$ that contains the elements of $\hat{\mathcal{N}}$ which does not contain any other subset of $\hat{\mathcal{N}}$, as follows:

$$\tilde{\mathcal{N}} = \left\{ Q \in \hat{\mathcal{N}} \mid \text{for all } Q' \in \hat{\mathcal{N}}, Q \neq Q' : Q' \not\subset Q \right\}.$$

Then, we have the decomposition of \hat{N} in the following theorem.

Theorem 5. The following decomposition of \hat{N} holds:

$$\hat{N} = \bigcup_{Q \in \tilde{\mathcal{N}}} \hat{N}(Q). \quad (11)$$

Proof. Since for all $p = 1, \dots, P$, we have:

$$N_p = \bigcup_{l=0}^{n+m} \bigcap_{Q \in \mathcal{F}^l} N_p(Q)$$

then, we get

$$\begin{aligned} \hat{N} &= \bigcap_{p=1}^P N_p \\ &= \bigcap_{p=1}^P \left(\bigcup_{l=0}^{n+m} \bigcap_{Q \in \mathcal{F}^l} N_p(Q) \right) \\ &= \bigcup_{l=0}^{n+m} \bigcap_{Q \in \mathcal{F}^l} \left(\bigcap_{p=1}^P N_p(Q) \right) \\ &= \bigcup_{l=0}^{n+m} \bigcap_{Q \in \mathcal{F}^l} \hat{N}(Q). \end{aligned}$$

□

Definition 7. A facet $F(Q)$, such that $Q \in \tilde{\mathcal{N}}$, is called *sorting set* for ML-MOLPP, we denote it by \mathcal{SP} .

4.2. Algorithm to generate the entire set \hat{N}

We can generate the entire set \hat{N} as follows: we begin by generate all the set N_p of non-dominated solutions of all the multi-objective problems of all the levels, after that we take their intersection. To do it suffices to apply the method of P.L. Yu and M. Zeleny described in [10], since each N_p in an union of facets, so, their intersection is not hard to get.

But the method proposed by P.L. Yu and M. Zeleny is composed of two major phases. The first in which the set of all extreme non-dominated points are determined and the second phase in which we recover the set of all the non-dominated points. In addition, the procedure must be repeated P times to obtain all the sets N_p , which makes the task very expensive in time and memory. For this reason, we propose another way to get the solution set \hat{N} .

4.2.1. Algorithm construction

1. First of all, we start by getting all the set \hat{N}^{dex} of non-dominated extreme points of all the MOLPPs of all levels. Using the procedure described by P.L. Yu and M. Zeleny, we generate all the sets N_p^{dex} , see [10, 9]. It means, for any $p = 1, \dots, P$, we use the P.L. Yu and M. Zeleny's algorithm to generate the set N_p^{dex} of all non-dominated extreme points of the problem 6, then we deduce the set \hat{N}^{dex} of non-dominated extreme points of ML-MOLPP.
2. We start looking for all the facets that are non-dominated for all levels at the same time. Note that if we assume that $\hat{N} \neq \emptyset$, then, we always find a non-dominated facet for a given ML-MOLPP. In the worst case, we get facets of dimension 0 (elements of \hat{N}^{dex}), or even segments (facet of dimension 1).
3. An important note is that we are looking for facets that have all their extreme points in the set \hat{N}^{dex} and those facets are of dimension less or equal to $n - 1$. These are the only facets that will pass the non-dominance test that we will set out in the next steps. So, to get those facets we do:
 - (a) Let \hat{d} denote the cardinal of \hat{N}^{dex} and \hat{N}^{dx_i} for $i = 1, \dots, \hat{d}$ denotes an element of \hat{N}^{dex} .
 - (b) For all $g = 1, \dots, \hat{d}$, we define the set of all subsets of the set $\{1, \dots, \hat{d}\}$ that contains exactly g elements, as follows:

$$\mathcal{U}^g = \left\{ U \subseteq \{1, \dots, \hat{d}\} : |U| = g \right\}. \quad (12)$$

- (c) For all $g = 1, \dots, \hat{d}$ and $U \in \mathcal{U}^g$, we can associate the convex hull of the points \hat{N}^{dx_i} for $i \in U$, as follows:

$$\mathcal{H}(U) = \left(\left\{ \hat{N}^{dx_i} \mid i \in U \right\} \right). \quad (13)$$

Note that the set $\mathcal{H}(U)$ is not always a facet of S .

- (d) Among all the sets of the form (13), we determinate those that are facets of S . In other words, determine a set of indexes $Q \subset M$ such as the facet $F(Q)$ verifies:

$$F(Q) = \mathcal{H}(U).$$

To this, simply substitute all the points \hat{N}^{dx_j} for $j \in U$, in the system $\tilde{A}x = \tilde{b}$, then take Q the set of indexes $i \in M$ for which:

$$\tilde{A}_i \hat{N}^{dx_j} = \tilde{b}_i \quad \text{for all, } j \in U.$$

If the obtained Q is an empty set, then $\mathcal{H}(U)$ is not a facet of S .

- (e) For all $g = 1, \dots, \hat{d}$ and $U \in \mathcal{U}^g$, define the set \mathcal{G}^g which contains only the faces of the feasible region S that can be defined as the convex hall of g extreme non-dominated points of \hat{N}^{dex} , as follows:

$$\mathcal{G}^g = \{F(Q) \mid F(Q) = \mathcal{H}(U)\}. \quad (14)$$

- (f) Since we are looking for a *sorting set* $F(Q)$ as a convex hall of some extremes non-dominated point ($F(Q) \in \mathcal{G}^g$), where $Q \in \mathcal{F}^l$, then from theorem 3, we assume that $g \geq n - l + 1$. Then, the set

$$\hat{\mathcal{G}}^g = \bigcup_{l \geq n-g+1} \{F(Q) \mid Q \in \mathcal{F}^l, F(Q) = \mathcal{H}(U)\} \quad (15)$$

contains only the facets of the feasible region S that can be defined as the convex hall of at least g -extreme non-dominated points of \hat{N}^{dex} and that need to be checked for non-dominance.

4. Define the set of all facets of S that needs to be checked for non-dominance by

$$\hat{\mathcal{G}} = \bigcup_{g=1}^{\hat{d}} \hat{\mathcal{G}}^g. \quad (16)$$

Note that

$$\hat{\mathcal{G}}^1 = \hat{N}^{dex}.$$

5. Use the following results to test the elements of $\hat{\mathcal{G}}$.

Proposition 1 (Non-dominance criterion for a point). *Let $\mathbf{P}^{dx} \in S$ and w_p be the optimal value of the problem (8) subject to*

$$\tilde{S}_p = \left\{ (x, v^p) \mid x \in S, c_p x - v^p \geq c_p \mathbf{P}^{dx}, v^p \geq 0 \right\}$$

and define

$$R = w_1 + \dots + w_p.$$

Then, \mathbf{P}^{dx} is a non-dominated point for ML-MOLPP, if and only if $R = 0$.

Theorem 6 (Non-dominance criterion for a facet). *Let $g \geq 2$ and $F(Q) \in \hat{\mathcal{G}}^g$, suppose that $\mathbf{P}^{dx} \in F(Q)$. Then: $\mathbf{P}^{dx} \in \hat{N} - \hat{N}^{dex}$ if and only if $F(Q) \subseteq \hat{N}$.*

Proof.

- (\Rightarrow) Since

$$\mathbf{P}^{dx} \in F(Q) \cap (\hat{N} - \hat{N}^{dex}),$$

then, $\mathbf{P}^{dx} \in F(Q)'$. From theorem 1, we have

$$\overline{F(Q)} = F(Q) \subseteq \hat{N}.$$

- (\Leftarrow) is immediate.

□

Remark 3. *Theorem 6 says that: if a given facet $F(Q)$ contains a non-dominated point P^{dx} in its relative interior $F(Q)'$, then, $F(Q)$ is a non-dominated facet for a ML-MOLPP. To apply the theorem, it suffices to choose g -numbers $\alpha_1, \dots, \alpha_g$ verifying*

$$0 < \alpha_t < 1 \quad \text{and} \quad \sum_{t=1}^g \alpha_t = 1$$

and check the point

$$P^{dx} = \sum_{t=1}^g \alpha_t N^{dx_t}$$

for non-dominance using proposition 1. In other words, a facet is non-dominated for a ML-MOLPP, if and only if, there is at least one non-dominated point that belongs to its relative interior.

6. During the execution of the procedure, if a non-dominated facet $F(Q)$ is found, then all the facets contained in $F(Q)$ must be discarded from the test, because they are all non-dominated facets, for this we do:
 - (a) For a non-dominated facet $F(Q) \in \hat{\mathcal{G}}^g$, denote by $\mathcal{V}_{F(Q)}$ the set of all the facets that are strictly contained in $F(Q)$.
 - (b) Denote by $\hat{\mathcal{S}}\mathcal{P}$, the set of all the sorting sets of the ML-MOLPP and for all $F(Q) \in \hat{\mathcal{S}}\mathcal{P}$ denote by \mathcal{W} the set of facets that will not be checked for the non-dominance test defined by

$$\mathcal{W} = \bigcup_{F(Q) \in \hat{\mathcal{S}}\mathcal{P}} \mathcal{V}_{F(Q)}.$$

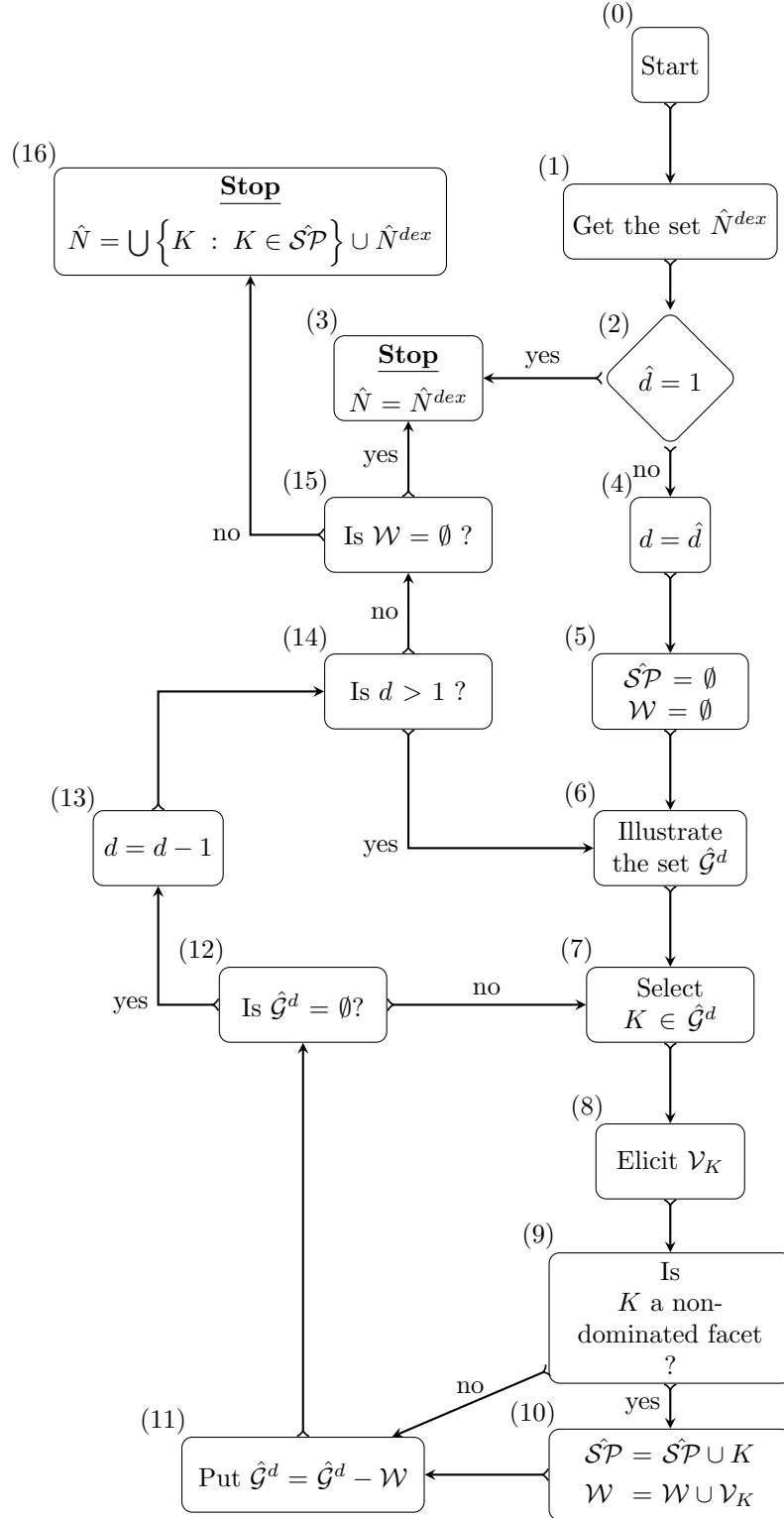
4.3. Full algorithm

In this section, we shall use the previous algorithm construction to describe a method for generating the set \hat{N} of all possible compromises of ML-MOLPP. Briefly, our method is first to generate the set \hat{N}^{dex} of common non-dominated extreme points between all the MOLPPs that constitute the ML-MOLPP. Then, test the non-dominance of all the facets of S that have all their extreme points on \hat{N}^{dex} .

Explanation of Flow Diagram

- Box 1** Generate the set \hat{N}^{dex} of extreme non-dominated points of a ML-MOLPP, by applying the method of P.L. Yu and M. Zeleny, see [10], to all the MOLPP that constitute the P -level problem. We get P -sets and then take their intersection.
- Box 6** Illustrate the set $\hat{\mathcal{G}}^d$, by locating all the facets of the feasible region S , that contains exactly d extreme non-dominated points that belong to \hat{N}^{dex} , see the previous section.
- Box (7)-(12)** Test all the elements of $\hat{\mathcal{G}}^d$, using the criterion given by the theorem 6. In step 10, we put the non-dominated ones in $\hat{\mathcal{S}}\mathcal{P}$ and put their non-dominated sub-facets in \mathcal{W} to avoid testing them again.

4.3.1. Flow Diagram



4.3.2. Algorithm

In this section, we present the full scheme of our method which we constructed previously and explained with the previous diagram.

Step 1 Get the set \hat{N}^{dex} as described in the previous section. If $\hat{d} = 1$, then stop with

$$\hat{N} = \hat{N}^{dex}.$$

Otherwise, go to **Step 2**.

Step 2 Put $\hat{\mathcal{S}}\mathcal{P} = \emptyset$ and $\mathcal{W} = \emptyset$.

Step 3 While $\hat{d} > 1$ do:

1. Set $\hat{d} = \hat{d} - 1$.
2. Illustrate the set $\hat{\mathcal{G}}^{\hat{d}}$.
3. Put $\hat{\mathcal{G}}^{\hat{d}} = \hat{\mathcal{G}}^{\hat{d}} - \mathcal{W}$.
4. **While** $\hat{\mathcal{G}}^{\hat{d}} \neq \emptyset$ **do**:
 - (a) Select $K \in \hat{\mathcal{G}}^{\hat{d}}$.
 - (b) Select $\mathbf{P}^{dx} \in K'$.
 - (c) Set $p = 1$.
 - (d) Solve the problem (8) subject to

$$\tilde{\mathcal{S}}_p = \left\{ (x, v^p) / x \in S, c_p x - v^p \geq c_p \mathbf{P}^{dx}, v^p \geq 0 \right\}$$

- (e) If $p > P$, put $R = w_1, \dots, w_P$ and go to step (f). Otherwise, put $p = p + 1$ and go to step (d).
- (f) If $R = 0$, elicit \mathcal{V}_K and put

$$\begin{aligned} \hat{\mathcal{S}}\mathcal{P} &= \hat{\mathcal{S}}\mathcal{P} \cup K, \\ \mathcal{W} &= \mathcal{W} \cup \mathcal{V}_K. \end{aligned}$$

- (g) Put $\hat{\mathcal{G}}^{\hat{d}} = \hat{\mathcal{G}}^{\hat{d}} - K$.
- (h) **End while**.

5. **End while**.

Step 4 Get the entire set \hat{N} as follows:

$$\hat{N} = \begin{cases} \bigcup \left\{ K : K \in \hat{\mathcal{S}}\mathcal{P} \right\} \cup \hat{N}^{dex}, & \text{if } \mathcal{W} \neq \emptyset \\ \hat{N}^{dex}, & \text{if } \mathcal{W} = \emptyset \end{cases}$$

5. Numerical example

Consider the following Two-levels-multi-objective linear programming problem:

Level 1

$$\begin{aligned} \max_{x_1} & (f_{11}(x) = 2x_1 + 2x_2, f_{12}(x) = -\frac{1}{2}x_1 + \frac{7}{25}x_2, f_{13}(x) = -\frac{1}{5}x_1 + \frac{1}{2}x_2), \\ & \text{such that } x_2 \text{ solves} \end{aligned} \tag{17}$$

Level 2

$$\max_{x_2} (f_{21}(x) = x_1 + 3x_2, f_{22}(x) = -2x_1 - 1x_2, f_{23}(x) = x_2)$$

subject to

$$x \in S = \{x \in \mathbb{R}^9 \mid Bx = b, x \geq 0, b \in \mathbb{R}^7\},$$

where

$$B = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & -4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 9 \\ 6 \\ 6 \\ -9 \\ 7 \\ 2 \end{pmatrix}.$$

Consider the following two MOLPPs:

$$(1) \begin{cases} \max & c_1 x \\ Bx & = b \\ x & \geq 0 \end{cases} \quad (2) \begin{cases} \max & c_2 x \\ Bx & = b \\ x & \geq 0 \end{cases}, \quad (18)$$

where

$$c_1 = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{7}{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 1 We apply the algorithm described in [10, 9], we get the following two non-dominated extreme points sets of the linear MOLPP (1) and (2):

- Non-dominated extreme points of problem (1):

$$N^{dx_1} = (6, 6), N^{dx_2} = (3, 6), N^{dx_3} = (1, 5).$$

- Non-dominated extreme points of problem (2):

$$N^{dx_1} = (6, 6), N^{dx_2} = (3, 6), N^{dx_3} = (1, 5),$$

$$N^{dx_4} = (0, 3), N^{dx_5} = (0, 0).$$

Then, we get

$$\hat{N}^{dex} = \{N^{dx_1}, N^{dx_2}, N^{dx_3}\}.$$

Step 2 Put

$$\hat{\mathcal{S}}\mathcal{P} = \emptyset \text{ and } \mathcal{W} = \emptyset.$$

Step 3 1. Choose

$$P^{dx} = \frac{N^{dx_1} + N^{dx_2} + N^{dx_3}}{3}.$$

By solving the linear problem (8) for all $p = 1, 2$, we get

$$R = w_1 + w_2 \neq 0.$$

Then, the set

$$\mathcal{H}(\{N^{dx_1}, N^{dx_2}, N^{dx_3}\})$$

is not non-dominated facet of the Two-Level-MOLPP .

- (a) $\hat{d} = \hat{d} - 1 = 2$

(b) The set $\hat{\mathcal{G}}^2$ is given by:

$$\hat{\mathcal{G}}^2 = \{ \mathcal{H}(\{N^{dx_1}, N^{dx_2}\}), \mathcal{H}(\{N^{dx_2}, N^{dx_3}\}) \},$$

see figure 3.

(c) Put

$$\hat{\mathcal{G}}^2 = \hat{\mathcal{G}}^2 - \mathcal{W} = \hat{\mathcal{G}}^2.$$

(d) i. Test the facet $K = \mathcal{H}(\{N^{dx_1}, N^{dx_2}\})$, choose

$$\mathbf{P}^{dx} = \frac{N^{dx_1} + N^{dx_2}}{2} \in K'.$$

By solving the linear problem (8) for all $p = 1, 2$, we get

$$R = w_1 + w_2 = 0.$$

Then, put

$$\hat{\mathcal{S}}\mathcal{P} = \{ \mathcal{H}(N^{dx_1}, N^{dx_2}) \}$$

and put

$$\hat{\mathcal{G}}^2 = \hat{\mathcal{G}}^2 - K \text{ and } \mathcal{W} = \emptyset.$$

ii. Test the facet $K = \mathcal{H}(\{N^{dx_2}, N^{dx_3}\})$, choose

$$\mathbf{P}^{dx} = \frac{N^{dx_2} + N^{dx_3}}{2} \in K'.$$

By solving the linear problem (8) for all $p = 1, 2$, we get

$$R = w_1 + w_2 = 0.$$

Then, put

$$\begin{aligned} \hat{\mathcal{S}}\mathcal{P} &= \hat{\mathcal{S}}\mathcal{P} \cup \{ \mathcal{H}(N^{dx_2}, N^{dx_3}) \} \\ &= \{ \mathcal{H}(N^{dx_1}, N^{dx_2}) \} \cup \{ \mathcal{H}(N^{dx_2}, N^{dx_3}) \} \end{aligned}$$

and put

$$\hat{\mathcal{G}}^2 = \hat{\mathcal{G}}^2 - K \text{ and } \mathcal{W} = \emptyset.$$

Step 4 Get the entire set \hat{N} of all possible compromises of the Two-Level-MOLPP (17), equal to:

$$\hat{N} = \mathcal{H}(N^{dx_1}, N^{dx_2}) \cup \mathcal{H}(N^{dx_2}, N^{dx_3}),$$

see figure 3.

Remark 4. Consider the non-dominated solution sets of the MOLPP (1)

$$N_1 = \mathcal{H}(N^{dx_1}, N^{dx_2}) \cup \mathcal{H}(N^{dx_2}, N^{dx_3}),$$

and the non-dominated solution set of the MOLPP (2)

$$N_2 = \mathcal{H}(N^{dx_1}, N^{dx_2}) \cup \mathcal{H}(N^{dx_2}, N^{dx_3}) \cup \mathcal{H}(N^{dx_3}, N^{dx_4}) \cup \mathcal{H}(N^{dx_4}, N^{dx_5}),$$

see Figure 4. Then, the set of all the possible compromises \hat{N} of the Two-levels-MOLPP (17) contains only the non-dominated facets of both problem (1) and (2), see Figure 3.

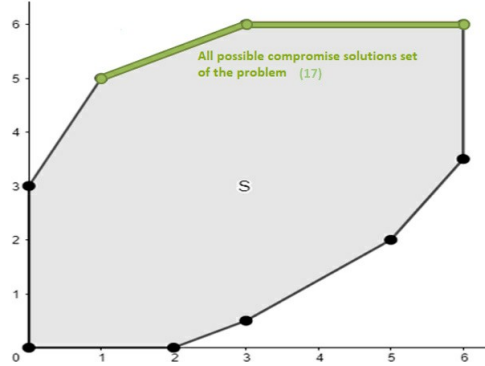


Figure 3: The set of all possible compromises of the Two-levels-MOLPP (17)

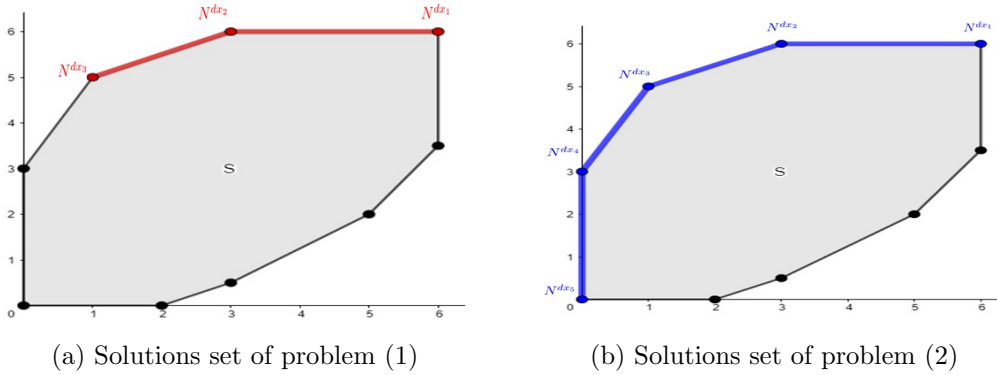


Figure 4: Non-dominated solutions sets of problems (1) and (2).

6. Conclusion

In this work, we construct a method that generalizes the P.L. Yu and M. Zeleny’s procedure to solve a ML-MOLPP. First, we defined the set of all the possible compromises of ML-MOLPP \hat{N} , it is a set containing all the non-dominated points of all the MOLPP that constitute the multi-level linear programming problem, also we defined the set of non-dominated extreme points. We give a criterion of non-dominance of facets. Then, we explain the construction of our method, we determine the facets that pass the test of non-dominance. After that, we describe the full algorithm that selects all non-dominated facets for our ML-MOLPP. The union of the obtained facets made up what we called set of all the possible compromises.

Our goal was to give a method that provides decision-makers involved in the decision-making process with a set of all the possible compromises that can be considered as the new feasible region on which the existing approaches can be applied to achieve a single satisfactory solution. We will try in the near future to examine the robustness of the method and report the results.

Finally, we illustrated our method with an example of two-levels-MOLPP. The extreme points are calculated manually and the linear mono-objective programs are solved on MATLAB R2007b after implementing the simplex method.

References

- [1] I.A. Baky. (2010). Solving multi-level multi-objective linear programming problems through fuzzy goal programming approach, *Applied Mathematical Modelling*, 34(9), 2377-2387. doi: [10.1016/j.apm.2009.11.004](https://doi.org/10.1016/j.apm.2009.11.004)
- [2] J. Bracken and J.T. McGill. (1973). Mathematical programs with optimization problems in the constraints, *Mathematical Programming and Its Applications*, 21(1), 37-44. url: <http://www.jstor.org/stable/169087>
- [3] J. Bracken and J.T. McGill. (1974). A method for solving mathematical programs with nonlinear programs in the constraints, *Operations Research*, 22(5), 1097-1101. doi: [10.1287/opre.22.5.1097](https://doi.org/10.1287/opre.22.5.1097)
- [4] S.Z. Delhoum, S. Radjef and F. Boudaoud. (2018). Generation of efficient and epsilon-efficient solutions in multiple objective linear programming, *Turk J Math*, 2018(42), 1031-1048. doi: [10.3906/mat-1702-12](https://doi.org/10.3906/mat-1702-12)
- [5] J. Lu, J. Han, Y. Hu and G. Zhang. (2016). Multi-level decision-making: A survey, *J. Inf. Sci*, 346-347, 63-87. doi: [10.1016/j.ins.2016.01.084](https://doi.org/10.1016/j.ins.2016.01.084)
- [6] C. Pieume, P. Marcotte, L. Fotso and P. Siarry. (2011). Solving Bi-level Linear Multi-objective Programming Problems, *American Journal of Operations Research*, 1(4), 214-219. doi: [10.4236/ajor.2011.14024](https://doi.org/10.4236/ajor.2011.14024)
- [7] S. Radjef and M.O. Bibi. (2011). An Effective Generalization of the Direct Support Method, *Mathematical Problems in Engineering*, 2011, 1-18. doi: [10.1155/2011/374390](https://doi.org/10.1155/2011/374390)
- [8] H.-S. Shih, Young-Jou Lai and E. Stanley Lee. (1996). Fuzzy approach for multi-level programming problems, *Computers and Operations Research*, 23(1), 73-91. doi: [10.1016/0305-0548\(95\)00007-9](https://doi.org/10.1016/0305-0548(95)00007-9)
- [9] P.L. Yu and M. Zeleny. (1974). The techniques of linear multi-objective programming. *RAIRO-Operations Research-Recherche Opérationnelle*, 8(V3), 51-71. MR: 439167 — Zbl: 0309.90028
- [10] P.L. Yu and M. Zeleny. (1975). The set of all non-dominated solutions in linear cases and a multi-criteria simplex method, *Journal of Mathematical Analysis and Applications*, 49(2), 430-468. doi: [10.1016/0022-247X\(75\)90189-4](https://doi.org/10.1016/0022-247X(75)90189-4)