# A Valuable Remark on Lipschitz in the First Variable Definition and System of Nonlinear Variational Inequalities

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**Abstract.** The goal of this paper is to present a critique on the incorrect use of the Lipschitz definition concerning the first variable and/or the second variable in the literature on the system of variational inequalities by many authors. The possible impact of this paper is rather important, it questions the results of different authors, particularly when taking into account that some of these papers are published in quite good mathematical journals. As a result, not only that the proofs are wrong, but also the credibility of the theorems themselves is compromised. In addition, this paper illustrate, using a counterexample, that there is an error in setting up first variable definition and the results obtained in listed references do not hold up in $H \times H$.

**Keywords**: Lipschitz continuous mappings, System of variational inequalities with different nonlinear mappings.

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## 1. Introduction

The first founder of the theory of variational inequalities was registered by Stampacchia in 1964. Since then, it has served as an interesting branch of applicable mathematics and engineering with a wide range of applications in physics, finance, social sciences, ecology, industry, and economics. It contains, as special cases: complementarity problems, systems of non-linear equations, problems of optimisation, and is also linked to problems of fixed points. A large class of problem in fluid mechanic, boundary value problem, transportation and equilibrium problems can be studied by variational inequalities which is another benefit of variational inequalities.

In recent years, various extensions and generalizations of variational inequalities to a system of variational inequalities have been considered and examined. Research on the approximate solvability of a class of a system of variational inequalities in a Hilbert space is due to Verma [19]. Since the 2004s the system is then extended by M Aslam Noor and some others to system of general variational inequalities [11], system of general mixed variational inequalities [12] and so on. There are a lot of papers written on System of variational inequalities, in all these publications the authors used an unclear Lipschitz continuous in the first variable and/or second variable definition. The aim of this paper is to illustrate that there is no sense in setting up this definition and all the results obtained in [1]–[25] have no benefit in $H \times H$.

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## 2. Preliminaries

Let $H$ be a real Hilbert space and $M$ be a nonempty closed and convex set in $H$, we denote by $\langle .,. \rangle$ and $\|.\|$, respectively the inner product and the induced norm in $H$.
In view of the fact that $T_1, T_2$ are both nonlinear operators, some researchers establish the problem of finding $(u^*, v^*) \in M \times M$ such that :

$$\begin{cases} \langle \alpha T_1 (v^*, u^*) + u^* - v^*, x - u^* \rangle \geq 0, \forall x \in M, \alpha > 0 \\ \langle \beta T_2 (u^*, v^*) + v^* - u^*, x - v^* \rangle \geq 0, \forall x \in M, \beta > 0 \end{cases} \tag{1}$$

which is called the system of nonlinear variational inequalities (see [13], [4],[11]). In the other hand, others (see [5], [22], [15]) establish a system of problems as follows:
Find $u^*, v^*, w^* \in H$ such that, for all $r, s, t > 0$,

$$\begin{cases} \langle \alpha T_1 (v^*, w^*, u^*) + u^* - v^*, x - u^* \rangle \geq 0, \forall x \in M, \alpha > 0, \\ \langle \beta T_2 (w^*, v^*, u^*) + v^* - w^*, x - v^* \rangle \geq 0, \forall x \in M, \beta > 0, \\ \langle \lambda T_3 (u^*, v^*, w^*) + w^* - u^*, x - w^* \rangle \geq 0, \forall x \in M, \lambda > 0. \end{cases} \tag{2}$$

For this purpose, they introduced the following definitions.

**Definition 1.** *A map $T : H \times H \to H$ is Lipschitz in the first variable if there exists a constant $\lambda > 0$ such that, for all pairs $x, y \in H$,*

$$\|T(x, u) - T(y, v)\| \leq \lambda \|x - y\|, \forall u, v \in H.$$

**Definition 2.** *A map $T : H \times H \times H \to H$ is Lipschitz in the first variable if there exists a constant $\lambda > 0$ such that, for all pairs $u, \grave{u} \in H$,*

$$\|T(u, v, w) - T(\grave{u}, \grave{v}, \grave{w})\| \leq \lambda \|u - \grave{u}\|, \forall v, \grave{v}, w, \grave{w} \in H.$$

By a careful reading, I discovered that Definition (1) or Definition (2) are the main tool of all papers. Also, I remarked that some authors have used the definition (1) implicitly. We shall take Huang and Noor [7] and Verma [19] as examples.

### 2.1. About Huang and Noor's paper [7] (see page 359)

Consider the following text taken from the proof of (Theorem 3.1 in [7]).
*proof*: First we need to evaluate $\|u_{n+1} - u^*\|$. From the nonexpansive property of the projection $P_K$ with (7) and (11), we have

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|(1 - \alpha_n) u_n + \alpha_n P_K [v_n - \rho T_1 (v_n, u_n)] - (1 - \alpha_n) u^* - \alpha_n P_K [v^* - \rho T_1 (v^*, u^*)]\| \\ &\leq (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \|P_K [v_n - \rho T_1 (v_n, u_n)] - P_K [v^* - \rho T_1 (v^*, u^*)]\| \\ &\leq (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \|[v_n - \rho T_1 (v_n, u_n)] - [v^* - \rho T_1 (v^*, u^*)]\| \\ &= (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \|v_n - v^* - \rho [T_1 (v_n, u_n) - T_1 (v^*, u^*)]\| . \end{aligned}$$

Since $T_1$ is **$\mu_1$-Lipschitzian in the first variable** and $(\gamma_1, r_1)$ $-$cocoercive, we have:

$$\begin{aligned} \|v_n - v^* - \rho [T_1 (v_n, u_n) - T_1 (v^*, u^*)]\|^2 &= \|v_n - v^*\|^2 - 2\rho \langle T_1 (v_n, u_n) - T_1 (v^*, u^*), v_n - v^* \rangle \\ &\quad + \rho^2 \|T_1 (v_n, u_n) - T_1 (v^*, u^*)\|^2 \\ &\leq \|v_n - v^*\|^2 + 2\rho\gamma_1 \|T_1 (v_n, u_n) - T_1 (v^*, u^*)\|^2 \\ &\quad - 2\rho r_1 \|v_n - v^*\|^2 + \rho^2 \|T_1 (v_n, u_n) - T_1 (v^*, u^*)\|^2 . \\ &\leq \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2 \mu_1^2\right] \|v_n - v^*\| .^2 \end{aligned}$$

### 2.2. About Verma's paper [19] (see page 207)

Let us look at the following cited text taken from the proof of (Theorem 2.1 in [12]): By applying Algorithm 2.1, we find

$$\begin{aligned}\|u_{k+1}-u^*\| &= \|(1-\alpha_k)\,u_k+\alpha_k P_K\left[v_k-\rho T\left(v_k,u_k\right)\right]-(1-\alpha_k)\,u^*-\alpha_k P_K\left[v^*-\rho T\left(v^*,u^*\right)\right]\| \\ &\le (1-\alpha_k)\,\|u_k-u^*\|+\alpha_k\left\|P_K\left[v_k-\rho T\left(v_k,u^k\right)\right]-P_K\left[v^*-\rho T\left(v^*,u^*\right)\right]\right\| \\ &\le (1-\alpha_k)\,\|u_k-u^*\|+\alpha_k\left\|v_k-v^*-\rho\left[T\left(v_k,u_k\right)-T\left(v^*,u^*\right)\right]\right\|.\end{aligned}$$

Since $T$ is **$\mu$-Lipschitz continuous in the first variable** and $(\gamma, r)$ −cocoercive, we have:

$$\begin{aligned}\|v_k-v^*-\rho\left[T_1\left(v_k,u_k\right)-T_1\left(v^*,u^*\right)\right]\|^2 &= \|v_k-v^*\|^2-2\rho\langle T\left(v_k,u_k\right)-T\left(v^*,u^*\right),v_k-v^*\rangle \\ &\quad+\rho^2\left\|T\left(v_k,u_k\right)-T\left(v^*,u^*\right)\right\|^2 \\ &\le \|v_k-v^*\|^2+2\rho\gamma\left\|T\left(v_k,u_k\right)-T\left(v^*,u^*\right)\right\|^2 \\ &\quad-2\rho r\,\|v_k-v^*\|^2+\rho^2\mu^2\,\|v_k-v^*\|^2. \\ &\le \left[1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2\right]\|v_k-v^*\|^2.\end{aligned}$$

## 3. Main Results

**Theorem 1.** *Let $T : H \times H \to H$, be $\lambda$-Lipschitzian in the first variable according to the definition (1), then there exists a $\lambda$-Lipschitzian function $g : H \to H$, such that for all $x \in H$:*

$$T(x,y)=g(x), \quad \forall y \in H.$$

**Proof**. Taking $y = x$ in definition (1), we find

$$\|T\left(x,u\right)-T\left(x,v\right)\|=0, \forall u,v \in H.$$

Therefore

$$T\left(x,u\right)=T\left(x,v\right), \forall u,v \in H.$$

Note that the value of $T(x, y)$ is always independently of the value of $y$. So there exists a $\lambda$-Lipschitzian function $g : H \to H$, such that for all $x \in H$,

$$T(x,y)=g(x), \quad \forall y \in H$$

□

**Corollary 1.** *Let $T : H^2 \to H$, be $\lambda$-Lipschitzian in the first variable according to the definition (1), then $T$ becomes an univariate mapping.*

## 4. Conclusion

Several authors have used the Lipschitz definition with respect to first variable and/or second variable to solve the system of nonlinear variational inequalities in Hilbert spaces. Unfortunately, they relied on incorrect definitions. The purpose of this paper is not to criticize the authors of the articles, but to examine what is wrong with their publications to help researchers who are interested to avoid these mistakes and pay attention when using references on system of nonlinear variational inequalities. Also, I show that there is no favor in setting up this definition and all the results obtained in [1]–[25] have no pregress in $H \times H$. We can redirect the previous studies [1]–[25] with the logical definitions as follow:

**Definition 3.** *A mapping $T : H \times H \to H$ is said to be $\lambda$-Lipschitz in the first variable if there exists constant $\lambda > 0$ such that, for all $u \in H$, for all pairs $x_1, x_2$ in $H$,*

$$\|T(x_1, u) - T(x_2, u)\| \leq \lambda \|x_1 - x_2\|.$$

**Definition 4.** *A mapping $T : H \times H \times H \to H$ is said to be $\lambda$-Lipschitz in the first variable if there exists constant $\lambda > 0$ such that, for all $u, v \in H$, for all pairs $x_1, x_2$ in $H$,*

$$\|T(x_1, u, v) - T(x_2, u, v)\| \leq \lambda \|x_1 - x_2\|.$$

## 5. Counterexample

For all $y \in \mathbb{R}$, it is clear that the function $(x, y) \to \cos(xy)$ is $|y|$-Lipschitzian in the first variable in the sense of Definition (3) :

$$\forall y \in \mathbb{R}, \forall (x, x') \in \mathbb{R}^2 : |\cos(xy) - \cos(x'y)| \leq |y||x - x'|.$$

But, if we applying the Definition (1) with $x = y = 1$ and $u = \frac{\pi}{2}$, $v = \frac{\pi}{4}$ we will find a contradiction:

$$|\cos 1.\frac{\pi}{2} - \cos 1.\frac{\pi}{4}| \leq |1 - 1| \Leftrightarrow |\cos\frac{\pi}{2} - \cos\frac{\pi}{4}| = 0 \Leftrightarrow \frac{\sqrt{2}}{2} = 0$$

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