Non-cooperative inventory games for defective items and quantity discounts using strategic complementarity

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Abstract. In this paper, the mathematical analysis for obtaining the equilibrium of the inventory games under strategic complementarities, the existence of defective items, and quantity discounts have been analyzed. The inventory system consists of many buyers who order a single type of product from one supplier. They compete with each other as a player in a non-cooperative game with strategic complementary. They maximize a supermodular payoff function and take into account some fraction of defective items from a lot of the arrival products. The concept of supermodular games is used to obtain the equilibria of these problems. A new existence theorem of Nash equilibrium in a specific condition has been proved. The optimum analysis has been justified for two conditions, that is the condition without discount and another without it. The numerical computations are provided using Python programming. At the end, the numerical result shows that elements of the Nash equilibrium set can be altered when discounts are considered. A quantity discount policy can be used by the supplier to prevent players from choosing the least Nash equilibrium.

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1. Introduction

The ordered games characterized by strategic complementarities are extension of the ordinary game. They are often defined in a lattice structure with specific ordering relations. Then, the class of games with strategic complementarities is called supermodular games. The study of these games was initiated by Topkis [21, 22] for submodular games. The properties and the equilibrium of the supermodular games are further developed by Milgrom and Roberts [17], Vives [24], and de' Orey [5]. Moreover, Milgrom and Shannon [18] provided an explanation of monotone comparative statics in supermodular games. Topkis [23] also completed his previous result, which was implemented in them. The concepts of supermodular games also have been applied to economy, stock competitions, and inventory problems, e.g., a study on the kinds of games using supermodularity by Amir [1]; NTU supermodular games in supermodular form by Koshevoy, et al. [12]; Stock competition problem using supermdular games by Chen [4], and some application results in inventory games. Focusing on the recent results in inventory problems, supermodular game concepts have been successfully implemented in the analysis of inventory games. For instance, Cachon [2], Cachon and Netessine [3], and Lippman and McCardle [15] analyzed a multiplayer inventory using this game concept.

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In the real condition of inventory management, some products with a defect will always be found after the screening process. The defects can be caused by an imperfect production and shipment processes. Therefore, the lot size that arrives at the buyer often contains defective items with a certain defective rate. In the deterministic case, the defective rate can be determined as a fraction of defective items. Based on the result by Jaber and Bonney [8], Jaber, et al. [9], and Konstantaras et al. [11], the defective rate conforms to a logistic curve. One of the learning curves used in the application is the S-shaped logistic curve. This value depends on a parameter representing how much the shipping process is carried out by the vendor. Another important assumption in inventory management relations is a synchronization policy. Using these policies, both vendor and the buyers can optimize the cycle time. Some research results about these issues in inventory management have been proposed by some authors. Hoque [6] proposed a comprehensive analysis of the synchronization process in the supply chain with a single manufacturer and multiple buyers under the integrated assumption. Jha and Shanker [10] analyzed the same topics for defective items under some additional assumptions such as a controllable lead time and service level constraint. Mandal and Giri [16] presented an analysis of a single vendor-multi-buyer integrated inventory using a synchronization process under controllable lead time and reduction of defective items. Lin [14] also proposed the analysis of a multiplayer inventory system using a synchronization process under controllable lead time and distribution-free demand. From those last three results, it can be concluded that the synchronization assumption works well in the inventory model for defective items.

One of the famous policies to promote the product in a supply chain coordination is a quantity discount. One of the results of supply chain analysis with discount policy is presented by Li and Liu [13]. Moreover, some researchers such as Wee [25] and Huang et. al. [7] have proposed the analysis of supply chain systems for deteriorating products with discounts. The handling process for deteriorating products is similar to the handling process for a defective item in inventory management. Therefore, it can be applied in the supply chain for defective items. Despite all efforts so far, we still haven't found any records of the inventory games application in a supply chain coordination with a discount policy.

To the date, there are have been no results concerning inventory games that employ strategic complementarity to tackle a combination problem to defective items and a quantity discount policy. This study has examined this issue within the framework of a multiplayer inventory system. The system consists of a single supplier of the product and multiple buyers as a player in a one-shot game. All the buyers' concern is on the defective items contained at arrival products. The existence of the defective items is in form of a fraction of defective items and follows a learning curve. Some important assumptions in inventory games are included, such as the synchronization process between one supplier and all buyers, wholesale contracts, buyback contracts, and quantity discounts. Quantity discounts are offered by the supplier in some promotion periods. Due to the strategic complementarity, NC-supermodular games (NC-SGs) are applied to solve the inventory games. This research is performed using an analytical method and a numerical representation using the dominance principle elements applied in Phyton programming. The results described in this paper have significance for the development of the application of the supermodular game theory, especially for inventory problems with defective items. The rest of the paper is organized as follows: the methodology and data used in this research are given in Section 2. The analysis of the games including formulation, optimum analysis, and numerical computations is explained in Section 3. Finally, some conclusions and remarks are presented in Section 4.

2. Methodology and data

In this research, a systematic literature review method is used to obtain information about how far the development of supermodular game concepts and their application in real problemsolving. The development of the concept of supermodular is still quite limited. Most of them are referred to in Topkis' s works [21, 22]. However, it found that the equilibrium existence theory can still be developed by adding some conditions to ensure the existence of two Nash equilibria. Moreover, the application of supermodular games in inventory problems is also restrictive for an inventory model with limited assumptions. Therefore, there is still an opportunity for research on the application of supermodular games in inventory games with various assumptions. Based on these literature analyses, the application of supermodular games on inventory games for defective items has never been researched before. Therefore, mathematical analysis methods including analytical and numerical approaches are applied to obtain the optimum result of inventory games for defective items and quantity discounts using supermodular games. For the analytical approach, some concepts in lattice theory and supermodular games have been used. The existence theorem of Nash equilibrium using some specific conditions is proposed. Finally, the appropriate simulation data is used for the numerical test.

3. Results and discussion

3.1. Assumptions and notations

This section presents an inventory system involving multiple buyers who receive a single type of product from one supplier. The supplier is responsible for producing the products and managing the shipping process to meet the total demand of all buyers. Each buyer aims to maximize their profits through non-cooperative game strategies. It is assumed that the game is delivered under strategic complementarities conditions. The supplier is not directly involved in the games. However, the supplier controlled the game by issuing some rules and contracts for the buyers. If all buyers accept these agreements, the game can be started. Three main contracts will be offered by the supplier. The details of these contracts are presented as follows:

1. *Synchronization process*. The production cycle of the supplier would be synchronized with the ordering cycles of the buyers. Such synchronization is useful to reduce the total related cost for the entire inventory system.
2. *Wholesale contract*. The supplier charges each buyer the amount price per unit purchased.
3. *Buyback contract*. The supplier charges the buyers amount of wholesale price but pays the buyer amount of price per unit remaining at the end of the cycle on each side of the buyer. The supplier also charges the buyers a standard cost for handling the remaining product return process.
4. *Quantity discounts*. In some promotion seasons, the supplier offers a quantity discount for their products when purchased in greater numbers. However, if this policy is applied, the supplier will not apply the buyback contract and will not provide a warranty cost for the defective items found after the inspection period.

When all the buyers accept the contracts from the supplier, the games will be started immediately. In the discount season, if all the buyers agree to purchase the product with a quantity discount, they will play the games with discount properties on the payoff function. The buyers will submit the order quantity $q = \sum_{i=1}^{n} q_i$ to the supplier before the selling season. The supplier then produces and delivers these products in a single shipment process. The buyers' concern is on the existence of defective items in the form of a fraction of defective items θ_i from a lot of sizes of arrival products on each buyer's s side. Thus, these defective items would be included in the inventory games scheme. The fraction of defective items follows an S-shaped logistic learning curve ([9],[11]) for one shipment process, that is $\psi(1) = \frac{a}{g+e^{1h}}$, where $a = 70.067$, $g = 819.76$, and $h = 0.7932$. Without loss of generality, we set the value of the fraction θ_i to be the same for each buyer. After receiving the new arrival products, all buyers run the inspection process with the inspection rate s_i^x and period $\frac{q_i}{s_i^x}$. All products will be considered non-defective items until they are detected in the inspection process. Once the inspection process is fully completed, the defective items will be stored temporarily until the next shipment arrives, resulting in holding costs. There are two distinct types of holding costs: one for defective items and another for non-defective items. These defective items are not part of the buyback contract. They will be returned to the supplier who will only pay the warranty cost ζ_i per unit of defective items for each buyer. Before going into any detail, the other mathematical notations are explained in the following table.

Notations	Explananation
D	Cumulative demand.
r_i	Retail price per unit product.
p_i	Purchasing cost per unit product (as wholesale price).
A_i	Ordering cost per unit product.
F_i	Freight cost per unit remaining product.
b_i	Buyback cost per unit of remaining product at the end of the cycle.
h_i^1	Holding cost per non-defective item per unit of time.
h_i^2	Holding cost per defective item per unit of time.
I_i^1	Inventory level for non-defective items.
I_i^2	Inventory level for defective items.
T_i	Transfer payment.
$u_i^j(q_i)$	Per unit material cost (IDR) as a function of q_i; j is the number of price breaks.
Φ_i	The buyer i' s payoff function.
q	Decision variable, a positive integer. Order quantity $q = \sum_{i=1}^{n} q_i$.
P	Production rate.
w	The supplier's sales price per unit product.
h_v	The supplier's holding cost per unit product per unit time for the supplier.
I_v	The supplier's inventory level for the supplier.
A_v	The supplier's setup cost.
c_v	The supplier's standard cost for a returning process.
Φ_v	The supplier's payoff function.

Table 1: *Mathematical Notations.*

The following sections will provide an analysis of inventory games without discounts as the initial discussion. This will be followed by an examination of games that include quantity discounts. Finally, a comparison of the results from these two analyses will be presented.

3.2. Game formulation

First, it assumed that every buyer plays as a player in the non-cooperative games using strategic complementarities and no quantity discount offered by the supplier. From this section onward, the buyer(s) is called by a player(s). The supplier as the coordinator will also use the results of these games as a basis for taking optimal decisions. Suppose that the feasible strategic space $S_i \subseteq \mathbb{R}$, $i \in \{1, \dots, n\}$ is a lattice with the usual ordering $\leq$ and *join* operation $x_i^{'} \vee x_i^{"} = \max\{x_i^{'}, x_i^{"}\}$ and *meet* operation $x_i^{'} \wedge x_i^{"} = \min\{x_i^{'}, x_i^{"}\}$, $x_i^{'}, x_i^{"} \in S_i$, such that the joint feasible strategic space for all players is formulated by $S = S_1 \times S_2 \times \cdots \times S_n$, which simply denoted by $S = \times_{i=1}^{n} S_i$. Set S is also a lattice with the component-wise order $\leq$ such that $\mathbf{x}^{'} \vee \mathbf{x}^{''} = \left(x_1^{'} \vee x_1^{"}, \dots, x_n^{'} \vee x_n^{"}\right)$ and $\mathbf{x}^{'} \wedge \mathbf{x}^{''} = \left(x_1^{'} \wedge x_1^{"}, \dots, x_n^{'} \wedge x_n^{"}\right)$, $\mathbf{x}^{'}, \mathbf{x}^{"} \in S$. Suppose

$\Phi_i : S \subset \mathbb{R}^n \to \mathbb{R}$ is payoff function for each buyer. When a selective joint strategy $\mathbf{x} \in S$ is played, then each player-i obtain their payoff $\Phi_i(\mathbf{x})$. For any selective joint strategy $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in S$, $\mathbf{x} \leq \mathbf{y}$ whenever $\Phi_i(\mathbf{x}) \leq \Phi_i(\mathbf{y})$. Furthermore, $\Phi_i : S \to \mathbb{R}$ is supermodular if for all $\mathbf{x}, \mathbf{y} \in S$,

$$\Phi_i(\mathbf{x}) + \Phi_i(\mathbf{y}) \leq \Phi_i(\mathbf{x} \wedge \mathbf{y}) + \Phi_i(\mathbf{x} \vee \mathbf{y}). \tag{1}$$

Let vector (y_i, x_{-i}) denotes the joint strategy vector with the strategy x_i of player i replaced by y_i in $\mathbf{x}$ and other components of $\mathbf{x}$ left unchanged. This notations also can be replaced by (q_i, q_{-i}) with the same meaning. All the buyers have one real valued payoff function Φ_i. These payoff is formed from two kinds of function, a reward function $g_i : S \to \mathbb{R}$, and a profit function $f_i : S \to \mathbb{R}$, such that $\Phi_i : S \to \mathbb{R}$, where $\Phi_i(q_i, q_{-i}) = g_i(q_i, q_{-i}) \cdot f_i(q_i, q_{-i})$. To elaborate on the explanation for a profit function, the inventory process in the supplier and the buyers will be explained as well as its respective costs.

The supplier carries out the production to fulfill the average demand from all buyers by following the synchronization process and under a finite production rate P $(P > D)$. Therefore, the average demand rate for each player i D_i is formulated by $D_i = \frac{q_i D}{q}$, where q_i is a replenishment quantity delivered to the player i every $\frac{q_i}{D_i}$ time units. The quantity of lot size from another player is denoted by q_{-i} which satisfies $q_{-i} = q - q_i$. The players place the same number of orders per unit of time and their order quantity lot size should be in proportion to their demand for shipment lot size. The shipment cycle time of the vendor is equal to the player's average ordering cycle time. The supplier makes one shipment process for all players simultaneously. Each buyer sells a single type of product in the selling season. After receiving the product and carrying out the inspection process, each player i will sell the non-defective product in the selling season. Each buyer knows well the information of their selling function. It is assumed that the selling function $L_i : S_i \to \mathbb{R}$ is real-valued. It is assumed that all buyers make a profit. The costs incurred as a result of the buyers' activities for the inventory management process comprise fixed ordering costs, fixed transportation costs, holding costs for non-defective items, holding costs for defective items, and transfer payment. The calculation of holding cost depends on the inventory level at the buyer's place. It takes a while until the product is sold, so the buyers need a budget for holding costs. There are two holding costs, holding costs for defective items and non-defective items. The holding cost term in the payoff function and the left inventory are calculated based on the on-hand inventory until the end of the cycle. On-hand inventory is explained in the following diagram (Figure 1)

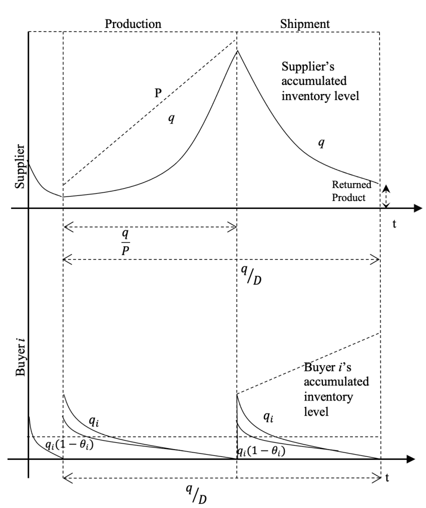

Figure 1: *Inventory Level.*

All incoming products are considered non-defective until any defects are identified at the conclusion of the inspection period. Suppose that there are $q_i\theta_i$ defective items in an arriving lot size q_i. Based on Figure 1, the average inventory level for defective items can be approximated by subtracting the accumulated defective items found during inspection time from the defective items throughout the cycle. Therefore, the average inventory level for defective items is

$$I_i^2 = \left(\frac{q_i^2(1-\theta_i)\theta_i}{D_i} - \frac{q_i^2\theta_i}{2s_i^x}\right). \tag{2}$$

The average inventory level per cycle for non-defective items is obtained from the sum of the average inventory level for non-defective items along the cycle and the defective items which are not detected yet before the end of the inspection time. Therefore, the average inventory level for non-defective items is

$$I_i^1 = \frac{q_i(1-\theta_i)}{D}\left(\frac{q_i^2\theta_i}{2s_i^x}\frac{D_i}{q_i(1-\theta_i)} + \frac{q_i(1-\theta_i)}{2}\right). \tag{3}$$

The last component of the profit function is a transfer payment for the left inventory at the end of the cycle in each buyer' s side. According to [19], the supplier should not make a profit from the remaining inventory in a news-vendor type problem. Therefore, the wholesale price per unit product must be greater than buyback cost per remaining unit, i.e.,$b_i \le p_i$. The transfer payment process is applied to the left product of non-defective items at the end of the cycle, i.e., $I_i^1 - L_i(q_i)$. The unit cost for these transfer payment is $h_i^1 + w_i - b_i$. It is assumed that the sales revenue is less than or equal to the inventory level for non-defective items, i.e. $L_i(q_i) \le R_i(q_i)$. To make a profit function, the revenue from selling season will be reduced by the costs. Therefore, the profit function is

$$f_i(q_i, q_{-i}) = \tfrac{D}{q(1-\theta_i)}\left(\left(p_i - h_i^1 - w_i\right)R_i(q_i) - \left(h_i^2 + w_i - \zeta_i\right)q_i^2\theta_i\left(\tfrac{1-\theta_i}{D_i} - \tfrac{1}{2s_i^x}\right)\right)$$

$$-\tfrac{D}{q(1-\theta_i)}\left(\left(A_i + F_i\right)q_i + \left(h_i^1 + w_i - b_i\right)\left(\tfrac{q_i^2(1-\theta_i)}{D}\left(\tfrac{D_i\theta_i}{2s_x^i(1-\theta_i)} + \tfrac{1-\theta_i}{2}\right) - L_i(q_i)\right)\right). \tag{4}$$

Now, we explain the second component of the payoff function, that is reward function. The reward is given by the supplier to all buyers. These reward is calculated based on the other players' left inventory for non-defective items in the end of the cycle and multiplied by some unit cost ε such that $g(q_i, q_{-i}) = \varepsilon\left(I_i^1 - L_i(q_i)\right)(q_{-i})$. Hence, the payoff function for each player is

$$\Phi_i(q_i, q_{-i}) = \varepsilon\left(I_i^1 - L_i(q_i)\right)\tfrac{D}{q(1-\theta_i)}\left(\left(p_i - h_i^1 - w_i\right)L_i(q_i) - \left(h_i^2 + w_i - \zeta_i\right)q_i^2\theta_i\left(\tfrac{1-\theta_i}{D_i} - \tfrac{1}{2s_i^x}\right)\right).$$

$$-\varepsilon\left(I_i^1 - L_i(q_i)\right)(q_{-i})\tfrac{D}{q(1-\theta_i)}\left(\left(A_i + F_i\right)q_i + \left(h_i^1 + w_i - b_i\right)\left(\tfrac{q_i^2(1-\theta_i)}{D}\left(\tfrac{D_i\theta_i}{2s_x^i(1-\theta_i)} + \tfrac{1-\theta_i}{2}\right) - L_i(q_i)(q_i)\right)\right). \tag{5}$$

All players play non-cooperative game using strategic complementarities performed in the supermodular game. These game can be notated as $G_m = (\rho, \{S_i\}_{i\in\rho}, \{\Phi_i\}_{i\in\rho})$, where $\rho = \{1, \ldots, n\}$ is number of players. A set of pure $\mathbf{x}^* \in S$ is called the Nash equilibrium of G_m if for each player i, the following condition is satisfied

$$\Phi_i\left(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*\right) \le \Phi_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_n^*\right). \tag{6}$$

To guarantee the existence of the equilibrium solutions of the games, it must be assumed that S_i is a nonempty compact lattice, $\Phi_i, i = \{1, \ldots, n\}$ is the supermodular function, and also upper-semicontinuous in y_i on $S_i(x_{-i})$. After all the buyers make their optimum decision, the supplier will use these values as a reference to make the optimal decision. Using the synchronization assumption, the optimal decision is applied to run the production process, shipment, and inventory management. The supplier's profit is obtained from the purchase revenue subtracted from several costs, covering setup costs, holding costs, internal shipment costs, and transfer costs. The setup and internal shipment costs are fixed, whereas the holding cost and the transfer cost depend on the supplier's inventory level and the left product on each supplier's side, respectively. The supplier's inventory level can be approximated by subtracting all players' accumulated inventory levels from the accumulated supplier inventory level as $I_v = \frac{q^2}{2P}$. Hence, the objective function of the supplier is

$$\Phi_v(q_i) = \left(wq + \sum_{i=1}^{n}(p_i - b_i)\left(I^1 - L_i(q_i)\right)\right) - \left(A_v + F_v + h_v\frac{\left(\sum_{i=1}^{n} q_i\right)^2}{2P} + \sum_{i=1}^{n} q_i\theta_i\zeta_i\right). \quad (7)$$

The second modeling is addressed to the condition of whether all players agree to purchase the product with a quantity discount offered by the supplier. According to Wee [25], an all-units quantity discount is assumed, and the material cost can be defined as follows

$$u_i^j(q_i^j) = \begin{cases} u_i^1, \ for \quad m_i^1 < q_i^1 \le m_i^2 \\ u_i^2, \ for \quad m_i^2 < q_i^2 \le m_i^3 \\ u_i^3, \ for \quad m_i^n < q_i^n. \end{cases} \quad (8)$$

The discounted prices for each player follow the established relationships $u_1^i > u_2^i \cdots > u_n^i$ and $m_1, m_2, \ldots, m_n$ stands for boundaries of the incremental quantities at state 1 to n. If all players choose to purchase an all-units quantity discount offered by the supplier, then the payoff function for all players will be changed. The unit material cost will be added to Equation (5). According to the contract offered by the supplier, if this policy is applied, the supplier will not apply the buyback contract and will not provide a warranty cost for the defective items after the inspection period. The reward function component given by the supplier is preserved. Hence, the player's payoff function for the discount condition is

$$\Phi_i(q_i, q_{-i}) = \varepsilon\left(I_i^1 - L_i(q_i)\right)(q_{-i})\frac{D}{q(1-\theta_i)}\left(\left(r_i - h_i^1 - p_i\right)u^i{}_j(q_i)L_i(q_i) - \left(h_i^2 + p_i\right)q_i^2\theta_i\left(\frac{1-\theta_i}{D_i} - \frac{1}{2s_i^x}\right)\right)$$

$$-\varepsilon\left(I_i^1 - L_i(q_i)\right)(q_{-i})\frac{D}{q(1-\theta_i)}\left(\left(A_i + F_i\right)q_i + \left(h_i^1 + p_i\right)\left(\frac{q_i^2(1-\theta_i)}{D}\left(\frac{D_i\theta_i}{2s_x^i(1-\theta_i)} + \frac{1-\theta_i}{2}\right) - L_i(q_i)\right)\right). \quad (9)$$

3.3. Optimum analysis and numerical examples

In this section, the optimum analysis has been analyzed using the standard optimization method. This analysis is focuses on how the conditions that ensure the existence of the Nash equilibrium of G_m. The Nash equilibrium is determined by the supermodular game theory presented by Topkis [23]. Furthermore, the Nash equilibrium of G_m also exists if some conditions of the player's strategic space and payoff function are held. These conditions are presented in the following main theorem:

Theorem 1. *Given a non-cooperative supermodular game $G_m = (\rho, \{S_i\}_{i\in\rho}, \{\phi_i\}_{i\in\rho})$ such that $S_i \subset \mathbb{R}^n, i = 1, \ldots, n$ is a nonempty compact lattice. If $\phi_i(y_i, x_{-i}), i = \{1, \ldots, n\}$ is continuous in y_i for each $x_{-i} \in S_{-i}$, and $\phi_i(x)$ can be formulated as $y_i \cdot H(x_{-i})$, $S_i, i = 1, \ldots, n$ is a real bounded interval, then the equilibrium set $\mathbf{x}^* = (x_i^*, \ldots, x_n^*)$ is a nonempty complete lattice. Furthermore, if x_{-i}^b is a greatest lower bound of the set of feasible strategies of all other players such that $H\left(x_{-i}^b\right) = 0$, which $H(x_{-i})$ is a real valued function defined in real bounded interval, then there is exists the greatest Nash equilibrium $x^{*\prime}$ and the least Nash equilibrium $x^{*\prime\prime}$.*

Proof. By following the proof by Topkis [23], it can be verified that G_m has two equilibria, the largest Nash equilibrium and the smallest Nash equilibrium. Because of $S_i \subset \mathbb{R}^n$, $i = 1, \ldots, n$ is a nonempty compact lattice, then $S = S_1 \times \cdots \times S_n$ is a nonempty compact lattice. For each $i = 1, \ldots, n, \phi_i(., x_{-i})$ is both upper semicontinuous and lower semicontinuous at the same time. There exist $\overline{y}_i \in S_i(x_{-i})$ such that $\phi_i(., x_{-i})$ hits a maximum at $\overline{y}_i$ such that $\overline{y}_i = argmax_{y_i \in S_i(x_{-i})} \phi(y_i, x_{-i})$. Moreover, there is $\underline{y}_i$ such that $\phi_i(., x_{-i})$ reaches a minimum point at $\underline{y}_i$ such that $\underline{y}_i = argmin_{y_i \in S_i(x_{-i})} \phi(y_i, x_{-i})$. Since G_m is supermodular game it holds that $\phi_i(y_i, x_{-i}), i = 1, \ldots, n$ is supermodular in y_i on S_i for each $x_{-i} \in S_{-i}$ and $\phi(y_i, x_{-i})$ have an increasing differences in (y_i, x_{-i}) for each $i = 1, \ldots n$. Furthermore, for each $x_{-i} \leq x'_{-i}$ and y'_i,

$$\phi_i\left(y_i, x'_{-i}\right) - \phi_i\left(y_i,\ x_{-i}\right) \leq \phi_i\left(y'_i, x'_{-i}\right) - \phi_i\left(y'_i, x_{-i}\right). \tag{10}$$

Based on Inequality (10), if other players can obtain a higher payoff according to their choice of strategy, then player i can obtain a higher payoff accordingly. If all opposing players ($n-1$ player) of player i do not choose a strategy $x_{-i} = x_i^b$, or there exist at most l opposing players ($l < n-2$) which play strategy $x_{-i} = x_{-i}^b$, then player i will choose $y_i = x_i^a, x_i \leq x_i^a, x_i \in S_i$ as an optimum response to obtain the higher payoff. Hence, $\left(x_i^a, x_{-i}^a\right)$ is the greatest Nash equilibrium. □

The form $\phi_i(y_i, x_{-i}) = y_i \cdot H(x_{-i})$ is an important condition to ensure that two Nash equilibrium exist or may the least equilibrium not exist. The term $H(x_{-i})$ must be presented in a single form of strategy without involving any constant terms. Since $S_i, i = 1, \ldots, n$ is a real bounded interval, i.e. $S_i \subset \mathbb{R}$, then the upper bound and the lower bound of S_i is the greatest and the least Nash equilibrium of G_m, respectively.

All known direct numerical methods to obtain the Nash equilibrium focus on the games with two-player only. Although some of them have been tried to apply in a game with three players, it is not easy to apply to every case and algorithm. One of the famous methods in a non-cooperative game is the dominance principle of the elements of the payoff matrix. It is designed for a noncooperative game with two players. Although has been developed for three players, it is still hard to compute. Accordingly, the numerical example is presented based on the games with two players (buyers) only. Because all players apply the strategy with complementarity properties, then the component-wise ordering, which is not a complete ordering, is still appropriate with the steps of the algorithms of the principle dominance. Although there exist pairs of strategies that can not be compared using component-wise ordering, e.g., (2,3) and (2,4), they can be ignored using the complementarity properties in a supermodular game. In these cases, if a pair of strategies $(q_1, q_2) = (2, 3)$ is being chosen, so a pair of strategies (2,4) will not be selected.

Next, some numerical illustrations of G_m for two players' cases are presented. Numerically, the Nash equilibrium for a class of non-cooperative games can be obtained using the dominance property principle of the payoff matrix. As explained in the previous section, the payoff method is commonly available for two-player cases only. This was also explained in detail by [20]. Furthermore, an algorithm can be developed in Python based on the pseudocode construction presented by [20]. Overall, the pseudocode uses a brute-force approach to find the Nash equilibrium points. Each combination of player strategies is examined to ensure that no player can improve their payoff by changing strategies. This method works well for games with a limited number of strategies; however, it becomes inefficient when players have numerous strategies due to the considerable computational time required. In our research, we developed an algorithm in Python based on the pseudocode by [20], as we found it suitable for the case we are discussing. The following is the algorithm in Python.

```
import numpy as np

# Define the bounds of the strategic space for Player I and Player II
player_I_strategies = np.array([1, 2, 3])  # Example strategies for Player I
player_II_strategies = np.array([1, 2, 3])  # Example strategies for Player II

# Payoff matrices for each player
payoff_player_I = np.zeros((len(player_I_strategies), len(player_II_strategies)))
payoff_player_II = np.zeros((len(player_I_strategies), len(player_II_strategies)))

# Input payoffs for each combination of strategies
for i, strategy_I in enumerate(player_I_strategies):
    for j, strategy_II in enumerate(player_II_strategies):
        # Manual input or using a function
        payoff_player_I[i, j] = int(input(f"Enter payoff for Player I at strategy ({strategy_I}, {strategy_II}): "))
        payoff_player_II[i, j] = int(input(f"Enter payoff for Player II at strategy ({strategy_I}, {strategy_II}): "))

# Determine Nash Equilibrium
nash_equilibrium = []

for i in range(len(player_I_strategies)):
    for j in range(len(player_II_strategies)):
        # Check if Player I gets the maximum payoff for the current strategy of Player II
        player_I_best_response = np.max(payoff_player_I[:, j]) == payoff_player_I[i, j]
        # Check if Player II gets the maximum payoff for the current strategy of Player I
        player_II_best_response = np.max(payoff_player_II[i, :]) == payoff_player_II[i, j]

        # If neither player can improve their payoff
        if player_I_best_response and player_II_best_response:
            nash_equilibrium.append((player_I_strategies[i], player_II_strategies[j]))

# Print the Nash Equilibrium results
if nash_equilibrium:
    print("Nash Equilibrium found at strategies:")
    for eq in nash_equilibrium:
        print(f"Player I: {eq[0]}, Player II: {eq[1]}")
else:
    print("No Nash Equilibrium found.")
```

Listing 1: *Python code to find Nash Equilibrium.*

Suppose that each player agrees to the following joint strategy space.

$$S = \{\mathbf{x} | q_1 = q_2, q_2 \in [1, 3)\} \cup ([3, 40] \times [3, 40]). \tag{11}$$

The sales function for the first and second buyers is defined by $L_i : S_i \to \mathbb{R}, i \in \{1, 2\}$, where $L_1(q_1) = K\left(\frac{3}{16}q_1^2 + \frac{1}{8}q_1\right)$, and $L_2(q_2) = K\left(\frac{3}{16}q_2^2 + \frac{1}{10}q_2\right)$, $K = \frac{1}{D}\left(\frac{D_i\theta_i}{s_i^x} + (1-\theta_i)^2\right)$. For numerical test, we consider the following the data: $D_1 = 45, D_2 = 35, s_1^x = 100$ unit product per unit time, $s_2^x = 95$ unit product per unit time, $p_1 = 400, p_2 = 450, A_1 = A_2 = 0.003, F_1 = F_2 = 0.001, h_1^1 = h_2^1 = 1.5, h_1^2 = h_2^2 = 2, w_1 = w_2 = 45, \zeta_1 = \zeta_2 = 1$, $b_1 = b_2 = 15$, and $\varepsilon = 0.5$. The fraction of the defective items for all buyers is the same value and expressed using an S-shaped logistic learning curve for one shipment process, i.e.

$\theta_i = \psi(1) = \frac{70.067}{819.76+e^{1.0.7932}} = 0.085, i = 1, 2$. First, if one of the player play a strategy $q_i \in [1, 3)$, then the other player must choose the same strategy. Therefore, a set $[1, 3)$ generate the Nash equilibrium of which $(1, 1)$ is the least Nash equilibrium. Suppose all buyers play a strategy contained in interval $[3, 40]$, then each player can be free to choose their strategy. The dominance principle of the payoff matrix is applied to determine the Nash equilibrium of G_m. For the numerical test, it takes 38 possible strategies for each player, e.g.,$q_i = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$ $31, 32, 33, 34, 35, 36, 37, 38, 39$, and 40. Therefore, the payoff matrix with 1444 elements is obtained. Using these data, we obtain the Nash equilibrium $(q_1^*, q_2^*) = (40, 40)$. Hence, the highest and the least Nash equilibrium is $(q_1^*, q_2^*) = (1, 1)$ and $(q_1^*, q_2^*) = (40, 40)$, respectively. If (q_1, q_2) selected by all players, the first and second buyers earn $\Phi_1 (40, 40) = 3713.033$ in 1000 IDR and $\Phi_2 (40, 40) = 4057.583$ in 1000 IDR as the payoff, respectively. The existence of the defective items still preserved the ordering relation in the strategic space. The supplier investigates the optimal decision by applying the synchronization principle. It means that the supplier takes $(q_1^*, q_2^*) = (40, 40)$ as the reference to determine his optimal decision, $q^* = 40 + 40 = 80$. Given the following data for the supplier (all values are in 1000 IDR): $A_v = 20, F_v = 1, h_v = 3$, and $P = 100$, then the largest and smallest profits for the supplier is $\Phi_v (1, 1) = 69.1$ in 1000 IDR, and $\Phi_v (40, 40) = 3872.29$ in 1000 IDR. Next, the numerical test is considered in the case when an all-units quantity discount is assumed. First, the unit material cost for each player is given by

$$u_1^j(q_1^j) = \begin{cases} u_1^1 = 400 \quad in \quad 1000IDR, \, for \quad 0 < q_1^1 \leq 15 \\ u_1^2 = 300 \quad in \quad 1000IDR, \, for \quad 15 < q_1^2 \leq 30 \\ u_1^3 = 200 \quad in \quad 1000IDR, \, for \quad 30 < q_1^3. \end{cases} \tag{12}$$

and

$$u_2^j(q_1^j) = \begin{cases} u_2^1 = 450 \quad in \quad 1000IDR, \, for \quad 0 < q_2^1 \leq 15 \\ u_2^2 = 350 \quad in \quad 1000IDR, \, for \quad 15 < q_2^2 \leq 30 \\ u_2^3 = 250 \quad in \quad 1000IDR, \, for \quad 30 < q_2^3. \end{cases} \tag{13}$$

The other parameters are used to illustrate the analytical results: $D_1 = 45, D_2 = 35, s_1^x = 100$ unit product per unit time, $s_2^x = 95$ unit product per unit time, $r_1 = 400, r_2 = 450, A_1 = A_2 = 0.003, F_1 = F_2 = 0.001, h_1^1 = h_2^1 = 1.5, h_1^2 = h_2^2 = 2, p_1 = p_2 = 45, \zeta_1 = \zeta_2 = 1$, and $\varepsilon = 0.5$. When a quantity discount is applied, both Player 1 and Player 2 will not choose $(q_1, q_2) = (1, 1)$ as a strategy profile. In otherwords, a strategy $(q_1, q_2) = (1, 1)$ is not Nash equilibrium. Based on the form of the payoff function in Equation (9), the higher the unit material cost, the greater payoff obtained by each player. Due to the use of strategic complementarity and the form of the unit material cost offered by the supplier for each player, then all players will purchase the product which is the third unit material cost, i.e.u_1^3 for Player 1 and u_2^3 for Player 2. Based on joint strategy space in Equation (11), $(q_1^*, q_2^*) = (40, 40)$ is selected as the Nash equilibrium. Hence, only a single Nash equilibrium would be obtained when a quantity discount is offered by the supplier. Each player will not purchase the product in a small number. Furthermore, it can be used as a new justification that the composition of Nash equilibrium of supermodular games can be altered using appropriate policy. In that case, a quantity discount policy can be applied by the supplier to prevent the players from choosing the least Nash equilibrium.

4. Conclusion

In this research, a new application of supermodular games for inventory games with strategic complementarities due to the existence of defective items has been proposed. All players are concerned about the existence of defective items which follow the logistic learning curve. According to analytical and numerical results, the concept of supermodular games can be used to obtain the optimum value of inventory games when the players use the strategic complementary, concerned about the existence of the defective item. To guarantee the presence of both the highest and lowest equilibrium, the reward function, which is based on the remaining inventory of the other players, can be utilized as a component of the buyers' payoff function. This result is also appropriate with our main existence theorem of the equilibrium. According to the numerical results, a quantity discount policy can be used by the supplier to prevent players from choosing the least Nash equilibrium. For further research, the same research can be conducted for inventory games with more complex assumptions and other policies.

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