A transient solution to the M/M/c queuing model equation with balking and catastrophes

Rakesh Kumar[1,*]

[1]*School of Mathematics, Shri Mata Vaishno Devi University, Katra, Jammu and Kashmir, India-182320*
E-mail: ⟨*rakesh.kumar@smvdu.ac.in*⟩

Abstract. In this paper, we consider a Markovian multi-server queuing system with balking and catastrophes. The probability generating function technique along with the Bessel function properites is used to obtain a transient solution to the queuing model. The transient probabilities for the number of customers in the system are obtained explicitly. The expressions for the time-dependent expected number of customers in the system are also obtained. Finally, applications of the model are also discussed.

Keywords: transient solution, M/M/c queuing system, catastrophes, generating function, balking

Received: August 03, 2016; accepted: December 08, 2017; available online: December 30, 2017

DOI: 10.17535/crorr.2017.0037

1. Introduction

Queuing models are playing an important role in modeling queuing situations in computer-communication networks, hospitals, supply chain management and in production processes. Recently, queuing models with catastrophes have attracted the attention of modelers. The occurrence of catastrophe leads to the annihilation of all units in a queuing system. Thus, modeling queuing systems with catastrophes is very important from the application point of view. To study the impact of noise bursts and virus on queues in computer networks, Chao [9] developed a queuing network model with catastrophes. He obtained the product-form solution of a queuing network model with catastrophes. Jain and Kumar [19] studied a queuing model with correlated arrivals, catastrophes and restoration along with its application in broadband communication networks. Di Crescenzo et al. [11] studied muscle contraction processes by using a single server queuing system with catastrophes. Jain and Kanethia [20] studied a queuing problem with catastrophes and environmental effects in connection with its application in studying insect populations. Krishna Kumar and Arivudainambi [21] studied an $M/M/1$ queuing system with catastrophes and derived its transient solution. Krishna Kumar et al. [22] obtained the transient

*Corresponding author.

solution to an $M/M/2$ queuing system with heterogeneous servers in the presence of catastrophes. He and Hu [14] studied an emergency supply chain multi-rescue system under uncertainties in large-scale disaster affected areas. They modeled it as a minimal queuing response time model of location and allocation. Janacek and Kvet [18] studied uncertainty in emergency systems using scenarios and fuzzy values. They compared the robust approach based on scenarios with the fuzzy approach. On the basis of a computational study, they found that the fuzzy approach performed better. Balking is a common feature of many practical situations. Balking is when a customer upon arriving sees a long queue and decides not to enter the queue. It has crucial effect on the performance of any system. The more balking there is, the lesser is the average number of customers in a system and hence lesser will be the revenue earned, if referring to revenue generating queuing systems. When we study a queuing system over a longer period of time, the characteristics of the system become independent of the initial conditions, i.e., their values become stationary. We say the system has reached steady-state. Solutions to queuing models which are time independent are known as steady-state solutions. In transient state or when time dependent, the initial conditions affect the behavior of the queuing system. The system characteristics vary with time. Transient solutions to queuing models are time dependent, for instance, they provide time-dependent probabilities for the number of customers in a system. Transient studies are more applicable than the steady-state studies, Whitt [42]. In many practical applications, the state of a system changes with time and such changes can be captured only by the transient behavior of the system. Business or service operations such as rental agencies or a physicians offices, which open and close, never operate under steady-state conditions. Hence, the investigation of the transient behavior of queuing systems is important from a theoretical as well as practical viewpoint. Owing to the practical applications of catastrophes and balking, we will study a multi-server Markovian queuing model with balking and catastrophes. We derive the transient solution to the model. The expression for a time-dependent expected number of customers in the system is also obtained. The remainder of the paper is arranged as follows: In Section 3, the queuing model is described. The mathematical model is formulated in Section 4. The transient solution to the model is obtained in Section 5. In Section 6, important results and practical applications of the model are presented. Finally, the paper provides conclusions in Section 7.

2. Literature review

The notion of a customers' impatience in queuing theory was introduced by researchers Haight ([15], [16]), Ancker and Gaffarian ([6], [7]), Obert [31], and Subba Rao ([36], [37]). Multi-server queuing systems incorporating customer impatience have found their way into many real life situations such as hospitals, computer-communications, retail stores, any many other situations. Montazer-Hagighi et al. [29], Abou-El-Ata and Hariri [3], Falin and Artalejo [13], Boots and Tijms [8], and Zohar et al. [45] studied and analyzed some multi-server queuing systems with balking and reneging. Yechiali [44] studied customer impatience in Markovian queuing systems with disaster and repair. He performed a steady-state analysis of these models. Altman and Yechiali [4] studied an infinite-server queue with system additional tasks and impatient customers. Shin and Choo [35] considered an $M/M/s$ queue with impatient customers and retrials. Al-Seedy et al. [5] studied an $M/M/c$ queue with balking and reneging, and derived its transient solution by using the probability generating function technique and the properties of the Bessel function. Ibrahim and Whitt [17] studied real-time delay estimation in overloaded multi-server queues with abandonments. Xiong and Altiok [43] studied multi-server queues with deterministic reneging times in connection with the timeout mechanism used in managing application servers in transaction processing environments. In the recent past, due attention has been given to the study of queuing systems with catastrophes. Krishna Kumar et al. [25] studied an $M/M/1$ queuing system with catastrophes, failures and repairs. They derived its transient solution using the continued fractions approach. The transient solution of a queuing system with catastrophes and state-dependent arrival and service rates was obtained by Kumar et al. [26]. Montazer-Hagighi [30] obtained the transient solution to a parallel multi-processor queuing system with task-splitting and feedback. Krishna Kumar and Pavai Madheshwari [24], Krishna Kumar et al. [23], and Dharmaraja and Kumar [10] used the probability generating function technique to obtain the transient solutions to queuing systems dealing with catastrophes. Paz and Yechiali [32] studied a single server Markovian queuing system in a random environment, where the system suffered from catastrophes leading to the destruction of all customers in the system. They studied the probabilistic behavior of a steady-state system. Using a probability generating function as well as via matrix geometric approach, Di Crescenzo et al. [12] studied birth-death process with catastrophes. They considered the first effective time occurrence. Tarabia [40] studied a single server Markovian queuing system with balking, catastrophes, server failures and repairs and derived its transient solution. Ammar [1] studied a queuing model with two heterogeneous servers queuing model subject to catastrophes and its application in a two-processor heterogeneous computer system. He derived a transient solution to the model. Sudesh [39] studied the transient behavior of a single server queue with disasters and impatient customers. Kumar and Sharma [27] considered the negative impact of customer impatience and incorporated the probability of retaining a reneging customer into an $M/M/1/N$ queuing system with reneging. They obtain a steady-state solution to the model. Kumar and Sharma [28] incorporated balking into a single server Markovian queuing system with retention of reneging customers. Vasiliadis [41] performed a transient analysis of an $M/M/k/N/N$ queuing system using a continuous time-homogeneous Markov system with finite capacity. Sudhesh et al. [38] performed a time-dependent analysis of quwuing with two-heterogenous servers and subject to disaster, repair, and customer impatience.

3. Queuing model assumptions

Arrivals into a queuing system occur subject to a Poisson process with intensity λ. Whenever the number of customers in the system is less than c, an arriving customer joins the system with a probability of one. On the other hand, an arriving customer may decide to join the queue with the probability of p and may balk with a probability of $1-p$ when the number of customers is c or more. There are c servers and service times at each server follow an exponential distribution with the parameter μ. Therefore, the mean service rate (μ_n) of the queuing system is given by

$$\mu_n = \begin{cases} n\mu, \ 0 \le n \le c-1 \\ c\mu, \ n \ge c \end{cases}$$

That is, if the number of customers in the system is less than the number of servers, they will be served at a mean rate of $n\mu$, and if the number of customers in the system is equal to or greater than the number of servers, they will be served at a mean rate of $c\mu$. The queue discipline is first-come-first-served (FCFS). Apart from arrival and service processes, when the system is not empty, catastrophes may also occur at the service facility as a Poisson process at the rate ψ. Whenever a catastrophe occurs in the system, all present customers are destroyed immediately, all the c servers become momentarily inactivated and the servers become ready for service immediately after the catastrophe.

4. Mathematical formulation of the model

In this section, we develop a mathematical model for the queuing problem under consideration. Let $\{X(t), t \ge 0\}$ be a stochastic process which represents the number of customers present in the system at time t.

Define, $P_n(t) = P[X(t) = n]$.

The following differential-difference equations are developed using birth-death arguments:

$$\frac{dP_0(t)}{dt} = -(\lambda + \psi)P_0(t) + \mu P_1(t) + \psi \tag{1}$$

$$\frac{dP_n(t)}{dt} = -\left(\lambda + \psi + n\mu\right) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad n = 1, 2, \ldots, c-1 \tag{2}$$

$$\frac{dP_c(t)}{dt} = -\left(\lambda p + \psi + c\mu\right) P_c(t) + \lambda P_{c-1}(t) + c\mu P_{c+1}(t) \tag{3}$$

$$\frac{dP_n(t)}{dt} = -\left(\lambda p + \psi + c\mu\right) P_n(t) + \lambda p P_{n-1}(t) + c\mu P_{n+1}(t), \quad n = c+1, \ldots \tag{4}$$

with $P_n(0) = \delta_{0n}$, the Kronecker delta symbol.

5. Transient solution of the model

We use the probability generating function technique to get the transient solution to the model.
The probability generating function $P(z,t)$ for the transient state probabilities $P_n(t)$ is defined by

$$P(z,t) = q_c(t) + \sum_{n=1}^{\infty} P_{n+c}(t)z^n;\ P(z,0) = 1 \tag{5}$$

where

$$q_c(t) = \sum_{n=0}^{c} P_n(t). \tag{6}$$

Adding the equations (1) − (3), we get

$$\frac{dq_c(t)}{dt} = -\xi\left[\sum_{n=0}^{c} P_n(t) - P_c(t)\right] + \xi - \lambda p P_c(t) - \xi P_c(t) + c\mu P_{c+1}(t)$$

$$\Rightarrow \frac{dq_c(t)}{dt} = -\lambda p P_c(t) + c\mu P_{c+1}(t) - \psi q_c(t) + \psi. \tag{7}$$

Now, multiplying the equation (4) by z^n and summing over the respective range of n, we obtain

$$\begin{aligned}\frac{d[\sum_{n=1}^{\infty} P_{n+c}(t)z^n]}{dt} &= \left[-(\lambda p + \psi + c\mu) + \left(\lambda p z + \frac{c\mu}{z}\right)\right]\sum_{n=1}^{\infty} P_{n+c}(t)z^n \\ &\quad + \lambda p z P_c(t) - c\mu P_{c+1}(t).\end{aligned} \tag{8}$$

Adding the equations (7) and (8) and using (5), we obtain

$$\begin{aligned}\frac{\partial P(z,t)}{\partial t} &= \left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right] P(z,t) \\ &\quad - \left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + c\mu)\right] q_c(t) \\ &\quad + \lambda p(z-1)P_c(t) + \psi.\end{aligned} \tag{9}$$

Solving the equation (9) by the Lagrangian method we get

$$\begin{aligned}P(z,t) &= \exp\left\{\left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right] t\right\} \\ &\quad + \int_0^t \left\{\lambda p(z-1)P_c(u) - \left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + c\mu)\right] q_c(u)\right\} \\ &\quad \times \exp\left\{\left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right](t-u)\right\} du \\ &\quad + \psi \int_0^t \exp\left\{\left[\left(\lambda p z + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right](t-u)\right\} du.\end{aligned} \tag{10}$$

If $\gamma = c\mu$, $\alpha = 2\sqrt{\lambda p\gamma}$ and $\beta = \sqrt{\frac{\lambda p}{\gamma}}$, then using the modified Bessel function of first kind $I_n(.)$ and the following Bessel function properties,

(i) $I_n(z) = I_{-n}(z)$

(ii) $I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z)$

(iii) $I_{n-1}(z) + I_{n+1}(z) = 2I_n^{'}(z)$

we get

$$\exp\left\{\left(\lambda pz + \frac{\gamma}{z}\right)t\right\} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t). \tag{11}$$

Using (11) in (10), we get

$$\begin{aligned} P(z,t) &= \exp\{[-(\lambda p + \psi + \gamma)]t\} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t) \\ &\quad + \lambda p \int_0^t P_c(u) \exp\{[-(\lambda p + \psi + \gamma)](t-u)\} \\ &\quad \times \sum_{n=-\infty}^{\infty} (\beta z)^n [\beta^{-1} I_{n-1}(\alpha(t-u)) - I_n(\alpha(t-u))] du \\ &\quad + \int_0^t q_c(u) \exp\{[-(\lambda p + \psi + \gamma)](t-u)\} \\ &\quad \times \sum_{n=-\infty}^{\infty} (\beta z)^n [-\lambda p \beta^{-1} I_{n-1}(\alpha(t-u)) \\ &\quad + (\lambda p + \gamma) I_n(\alpha(t-u)) - \gamma\beta I_{n+1}(\alpha(t-u))] du \\ &\quad + \psi \int_0^t \exp\{[-(\lambda p + \psi + \gamma)](t-u)\} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-u)) du. \end{aligned} \tag{12}$$

As $P(z,t) = q_c(t) + \sum_{n=1}^{\infty} P_{n+c}(t) z^n$, we compare the coefficients of z^n on either side of (12), and we obtain for $n = 1, 2, \ldots$

$$\begin{aligned} P_{n+c}(t) &= \exp\{-(\lambda p + \psi + \gamma)t\} \beta^n I_n(\alpha t) + \lambda p \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} \\ &\quad \times \left[I_{n-1}(\alpha(t-u))\beta^{n-1} - I_n(\alpha(t-u))\beta^n\right] P_c(u) du \\ &\quad - \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} q_c(u) [\lambda p I_{n-1}(\alpha(t-u))\beta^{n-1} \\ &\quad - (\lambda p + \gamma) I_n(\alpha(t-u))\beta^n + \gamma I_{n+1}(\alpha(t-u)\beta^{n+1}] du \\ &\quad + \psi \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} \beta^n I_n(\alpha(t-u)) du. \end{aligned} \tag{13}$$

Comparing the terms free of z on either side of equation (12), that is, for $n = 0$, we get

$$\begin{aligned} q_c(t) &= \exp\{-(\lambda p + \psi + \gamma)t\} I_0(\alpha t) + \lambda p \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} P_c(u) \\ &\quad \times \left[I_1(\alpha(t-u))\beta^{-1} - I_0(\alpha(t-u)) \right] du \\ &\quad - \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} q_c(u) \\ &\quad \times \left[\alpha I_1(\alpha(t-u)) - (\lambda p + \gamma) I_0(\alpha(t-u)) \right] du \\ &\quad + \psi \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} I_0(\alpha(t-u)) du. \end{aligned} \tag{14}$$

As $P(z,t)$ does not contain terms with negative powers of z, the right hand side of (13) with n replaced by $-n$, must be zero. Thus, we obtain

$$\begin{aligned} &\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} q_c(u) \, [\lambda p I_{n+1}(\alpha(t-u))\beta^{n-1} - (\lambda p + \gamma) I_n(\alpha(t-u))\beta^n \\ &\qquad + \gamma I_{n-1}(\alpha(t-u))\beta^{n+1}] du \end{aligned}$$

$$\begin{aligned} &= \exp\{-(\lambda p + \psi + \gamma)t\} I_n(\alpha t)\beta^n + \lambda p \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} P_c(u) \\ &\quad \times [I_{n+1}(\alpha(t-u))\beta^{n-1} - I_n(\alpha(t-u))\beta^n] du \\ &\quad + \psi \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} I_n(\alpha(t-u))\beta^n du. \end{aligned} \tag{15}$$

Using (15) in (13) we get for $n = 1, 2, \ldots$

$$P_{n+c}(t) = n\beta^n \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} \frac{I_n(\alpha(t-u))}{(t-u)} P_c(u) du. \tag{16}$$

The remaining probabilities $P_n(t), n = 0, 1, \ldots, c$ can be obtained by solving the equations (1) and (2). In matrix form, the equations (1) and (2) can be written as:

$$\frac{d\mathbf{P(t)}}{dt} = A\mathbf{P(t)} + \gamma P_c(t)\mathbf{e_1} + \psi \mathbf{e_2}. \tag{17}$$

where the matrix $A = (a_{i,j})_{c \times c}$ is given as:

$$A = \begin{bmatrix} -(\lambda + \psi) & \mu & \cdots & 0 \\ \lambda & -(\lambda + \psi + \mu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (c-1)\mu \\ 0 & 0 & \cdots & -(\lambda + \psi + (c-1)\mu) \end{bmatrix}$$

$\mathbf{P(t)} = (P_0(t) \ \ P_1(t) \ \ \ldots \ \ P_{c-1}(t))^T$, $\mathbf{e_1} = (0 \ \ 0 \ \ \ldots \ \ 1)^T$ and $\mathbf{e_2} = (1 \ \ 0 \ \ \ldots \ \ 0)^T$ are column vectors of order c.
Let $\mathbf{P^*(s)} = (P_0^*(s) \ \ P_1^*(s) \ \ \ldots \ \ P_{c-1}^*(s)])^T$ denotes the Laplace transform of $\mathbf{P(t)}$. Taking the Laplace transform of equation (17) and solving for $\mathbf{P^*(s)}$, we get

$$\mathbf{P^*(s)} = (sI - A)^{-1}\{\gamma P_c^*(s)\mathbf{e_1} + \mathbf{P(0)} + \frac{\psi}{s}\mathbf{e_2}\} \tag{18}$$

with $\mathbf{P(0)} = (1 \ \ 0 \ \ \ldots \ \ 0)^T$. Thus, only $P_c^*(s)$ remains to be found. We observe that if $\mathbf{e} = (1 \ \ 1 \ \ \ldots \ \ 1)_{c\times 1}^T$, then

$$\mathbf{e}^T\mathbf{P^*(s)} + P_c^*(s) = q_c^*(s). \tag{19}$$

Define

$$f(s) = \left[(s + \lambda p + \gamma + \psi) - \sqrt{(s + \lambda p + \gamma + \psi)^2 - \alpha^2}\right].$$

Taking the Laplace transform of (14) and solving for $q_c^*(s)$, we obtain

$$s(s + \psi)q_c^*(s) = (s + \psi) + sP_c^*(s)\frac{1}{2}[f(s) - \alpha\beta]. \tag{20}$$

Using equation (20) in (19) and simplifying, we get

$$P_c^*(s) = \left(\frac{s + \psi}{s}\right) \times$$

$$\frac{1 - s\mathbf{e^T}(sI - A)^{-1}(\mathbf{P(0)} + \frac{\psi}{s}\mathbf{e_2})}{\{(s + \lambda p + \psi) - \frac{1}{2}[f(s)] + (s + \psi)\gamma\mathbf{e^T}(sI - A)^{-1}\mathbf{e_1}\}}. \tag{21}$$

In equations (18) and (21), $(sI - A)^{-1}$ has to be found. Let us assume that

$$(sI - A)^{-1} = (b_{ij}^*(s))_{c\times c}$$

We note that $(sI - A)^{-1}$ is almost lower triangular. Following Raju and Bhat [33], we obtain, for $i = 0, 1, \ldots, c - 1$

$$b_{ij}^*(s) = \begin{cases} \frac{1}{(j+1)\mu}\frac{u_{c,j+1}(s)u_{i,0}(s) - u_{i,j+1}(s)u_{c,0}(s)}{u_{c,0}(s)}, & j = 0, 1, \ldots, c - 2. \\ \frac{u_{i,0}(s)}{u_{c,0}(s)}, & j = c - 1. \end{cases} \tag{22}$$

where $u_{i,j}(s)$ are recursively given as

$$u_{i,i} = 1, \qquad i = 0, 1, \ldots, c - 1.$$

$$u_{i+1,i} = \frac{s + \lambda + \psi + i\mu}{(i+1)\mu}, \qquad i = 0, 1, \ldots, c - 2.$$

$$u_{i+1,i-j} = \frac{(s + \lambda + \psi + i\mu)u_{i,i-j} - \lambda u_{i-1,i-j}}{(i+1)\mu}, \ j \leq i, i = 1, 2, 3, \ldots, c - 2.$$

$$u_{c,j} = \begin{cases} [s+\lambda+\psi+(c-1)\mu]u_{c-1,j} - \lambda u_{c-2,j}, & j = 0,1,\ldots,c-2. \\ s+\lambda+\psi+(c-1)\mu, & j = c-1. \end{cases} \quad . \tag{23}$$

and
$u_{i,j} = 0$, for other i and j. Using these in equation (21), we get

$$P_c^*(s) = \left(\frac{s+\psi}{s}\right) \times$$

$$\frac{\{1-(s+\psi)\sum_{i=0}^{c-1} b_{i,0}^*(s)\}}{\{(s+\lambda p+\psi) - \frac{1}{2}[f(s)] + (s+\psi)\gamma\sum_{j=0}^{c-1} b_{j,c-1}^*(s)\}} \tag{24}$$

and for $k = 0,1,\ldots,c-1$ from equation (18), we get

$$P_k^*(s) = \left(1+\frac{\psi}{s}\right) b_{k,0}^*(s) + \gamma b_{k,c-1}^*(s) P_c^*(s) \tag{25}$$

We observe that $b_{i,j}^*(s)$ are all rational algebraic functions in s. The cofactor of the $(i,j)^{th}$ element of $(sI-A)$ is a polynomial of degree $c-1- \mid i-j \mid$. Since the characteristic roots of A are all distinct, the inverse transform $b_{i,j}(t)$ of $b_{i,j}^*(s)$ can be obtained by partial fraction decomposition. Let $s_i, i = 0,1,\ldots,c-1$, be the characteristic roots of the matrix A. Then after partial fraction decomposition and simplification [34], $P_c^*(s)$ equals to

$$\frac{\left(1+\frac{\psi}{s}\right) E^*(s)}{\frac{1}{2}[f(s)]\left[1 - \frac{2\gamma(1-F^*(s))}{(s+\lambda p+\gamma+\psi) - \sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}}\right]}. \tag{26}$$

where

$$E^*(s) = \sum_{i=0}^{c-1} \frac{E_i}{s-s_i} \tag{27}$$

$$F^*(s) = \sum_{i=0}^{c-1} \frac{F_i}{s-s_i} \tag{28}$$

with constants E_i and F_i given by

$$E_i = \lim_{s\to s_i}(s-s_i)\left[1 - \sum_{l=0}^{c-1}(s+\psi)b_{l,0}^*(s)\right] \tag{29}$$

$$F_i = \lim_{s\to s_i}(s-s_i)\left[\sum_{l=0}^{c-1}(s+\psi)b_{l,c-1}^*(s)\right] \tag{30}$$

Hence,(26) simplifies into

$$P_c^*(s) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n+1)\frac{(-1)^m}{\gamma}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}\binom{n}{m}\left[\left(1+\frac{\psi}{s}\right)E^*(s)(F^*(s))^m \frac{[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]^{n+1}}{(n+1)\alpha^{n+1}}\right]. \quad (31)$$

Taking Laplace inverse of (31), we obtain

$$P_c(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\gamma}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}\binom{n}{m}\left[\int_0^t E(t-u)\int_0^u F^{C(m)}(u-v) \exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv \right. \quad (32)$$
$$\left. +\psi\int_0^t G(t-u)\int_0^u F^{C(m)}(u-v)\exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv\right].$$

[*Using* $L^{-1}\left[\frac{[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]^{(n+1)}}{\{\alpha^{n+1}\}}\right] = \frac{e^{-\alpha t}I_{n+1}(\alpha t)}{t}$, $L^{-1}(E^*(s)F^*(s)) = \int_0^t E(u)F(t-u)du$ *(Convolution theorem),* $F^{C(m)}(t)$ *is m-fold convolution of* $F(t)$ *with itself with* $F^{C(0)} = \delta(t)$*,* I^{st} *we take convolution between E and F and then between F and* $\frac{[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]^{(n+1)}}{\{\alpha^{n+1}\}}$.]

$$P_c(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{c\mu}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}\binom{n}{m}\left[\int_0^t E(t-u)\int_0^u F^{C(m)}(u-v) \exp\{-(\lambda p+c\mu+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv \right. \quad (33)$$
$$\left. +\psi\int_0^t G(t-u)\int_0^u F^{C(m)}(u-v)\exp\{-(\lambda p+c\mu+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv\right].$$

where $G(t) = \int_0^t E(u)du$ and $F^{C(m)}(t)$ is $m-fold$ convolution of $F(t)$ with itself with $F^{C(0)} = \delta(t)$, the Dirac delta function, $E(t-u)$ is a function of u at a particular value of t which is obtained from the convolution of two functions $E(t)$ and $F(t)$, where $E(t) = L^{-1}(E^*(s))$ and $F(t) = L^{-1}(F^*(s))$.

Now, the Laplace inverse of equation (25) yields,

$$P_k(t) = b_{k,0}(t)+\psi\int_0^t b_{k,0}(u)du+\gamma\int_0^t b_{k,c-1}(u)P_c(t-u)du, k = 0,1,\ldots,c-1. \quad (34)$$

or

$$P_k(t) = b_{k,0}(t) + \psi \int_0^t b_{k,0}(u)du + c\mu \int_0^t b_{k,c-1}(u)P_c(t-u)du, k = 0, 1, \ldots, c-1. \tag{35}$$

where $P_c(u)$ is given by (33). Thus, the equations (16), (33) and (35) determine all the state probabilities explicitly.

6. Results

In this section we present the results of the paper. First, the explicit time-dependent probabilities of the number of customers in the system are provided, and then the expression for the expected number of customers in the system is given.

(i) The mathematical model described by the equations (1) − (4) was solved in the previous section in transient state using the probability generating function technique. The explicit time-dependent probabilities of the number of customers in the system are obtained in equations (16), (33) and (35) respectively as given below:

$$P_k(t) = b_{k,0}(t) + \psi \int_0^t b_{k,0}(u)du + c\mu \int_0^t b_{k,c-1}(u)P_c(t-u)du, k = 0, 1, \ldots, c-1$$

$$P_{n+c}(t) = n\beta^n \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\} \frac{I_n(\alpha(t-u))}{(t-u)} P_c(u)du; n = 1, 2, ...$$

$$P_c(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{c\mu} \left(\frac{\alpha}{2\lambda p}\right)^{n+1} \binom{n}{m} \left[\int_0^t E(t-u) \int_0^u F^{C(m)}(u-v) \exp\{-(\lambda p + c\mu + \psi)v\} \frac{I_{n+1}(\alpha v)}{v} dudv + \psi \int_0^t G(t-u) \int_0^u F^{C(m)}(u-v) \exp\{-(\lambda p + c\mu + \psi)v\} \frac{I_{n+1}(\alpha v)}{v} dudv\right]$$

where $G(t) = \int_0^t E(u)du$ and $F^{C(m)}(t)$ is $m - fold$ convolution of $F(t)$ with itself with $F^{C(0)} = \delta(t)$, the Dirac delta function, $\gamma = c\mu$ and $\alpha = 2\sqrt{\lambda p \gamma}$.

(ii) Expected number of customers in the system, E[X(t)]
Using the state probabilities obtained in equation (16), (33) and (35), the expected system size is obtained as follows

$$E[X(t)] = M(t) = m(t) + r(t) = \sum_{n=1}^{c-1} nP_n(t) + \sum_{n=c}^{\infty} nP_n(t) \tag{36}$$

Differentiating (36) with respect to t we get

$$M'(t) = m'(t) + r'(t) = \sum_{n=1}^{c-1} nP_n'(t) + \sum_{n=c}^{\infty} nP_n'(t) \tag{37}$$

Multiplying (1)-(3) by n and summing over the range of n, we get

$$M'(t) = -\psi M(t) + \lambda \sum_{n=0}^{c-1} P_n(t) - \mu m(t) + (\lambda p - c\mu) \sum_{n=c}^{\infty} P_n(t) \tag{38}$$

The above equation is of the form $y' + Py = Q$ whose solution is

$$\begin{aligned} M(t) = (\lambda p - c\mu) \sum_{n=c}^{\infty} \int_0^t P_n(u) exp(-\psi(t-u)) du + \\ \lambda \sum_{n=0}^{c-1} \int_0^t P_n(u) exp(-\psi(t-u)) du - \\ \mu \int_0^t m(u) exp(-\psi(t-u)) du \end{aligned} \tag{39}$$

The equation (39) gives the expression for time-dependent expected number of customers in the system.

6.1. Applications of the model

In this subsection we discuss the possible application of our model.

1. **In Call Center Management**
   A call center is a centralized office used for receiving or transmitting a large volume of requests by telephone. It has an open work place for call center agents, with work stations (computers) and a telephone set. A call center can be viewed as a queuing system with incoming requests (enquires) as customers and various agents with work stations as servers. More precisely, it forms a multi-server queuing system. A customer who finds the server busy many times may get impatient and discontinue their call. Such a customer can be considered as a balked customer. Moreover, with the occurrence catastrophic events like power failure, virus attacks, and corruption of hard discs of computer-systems at call centers, all the calls will get lost and all the servers will become inactivated. Thus, a multi-server queuing system with catastrophes can be used for modeling the performance analysis of call centers .

2. **In Emergency Department of Hospitals**
   Hospitals offer various kinds of services such as such as OPD (Pediatrics, Neurology, General Medicine, Cardiology, Dermatology, ENT), IPD, Diagnostic

testing through Pathology labs, Diet counseling, and Emergency health care services. In almost every hospital there is a dedicated emergency department. This department handles the critical patients requiring immediate attention. If the service to each patient is not available immediately they wait in queue for their turn. Thus, an emergency department in hospitals can be viewed as a queuing system with patients as waiting customers, and the different doctors as multiple servers. An incoming patient who sees the system congested may decide not to join it and goes somewhere else for the service. Such patients can be considered as balked customers. Further, due to a power failure, lack of oxygen supply, fire etc., a catastrophic event may occur in the emergency department resulting in all the patients dying. The model investigated in this paper can be applied to study the performance of emergency departments prone to catastrophes.

7. Conclusions

A Markovian multi-server queuing model with balking and catastrophes was studied. The transient solution of the model is obtained by using a probability generating function technique, and properties of the Bessel function. Further, the expression for time-dependent expected number of customers in the system is also derived. The queuing model considered in this paper has its powerful application in modeling emergency situations in hospitals and managing call centers.

References

[1] Ammar, S.I. (2014). Transient behavior of a two-processor heterogenous system with catastrophes, server failure and repairs. Applied Mathematical Modeling, 38, 2224–2234.

[2] Abramowitz, M., Stegun, I.A. (1970). Handbook of mathematical functions. Dover, New York.

[3] Abou-El-Ata, M.O. and Hariri, A.M.A. (1992). The M/M/c/N queue with balking and reneging. Computers and Operations Research, 19(8), 713–716.

[4] Altman, E. and Yechiali, U. (2008) Infinite-server queues with systems additional tasks and impatient customers. Probability in the Engineering and Informational Sciences, 22(4), 477–493.

[5] Al-Seedy, R.O., El-Sherbiny, A.A., El-Shehawy, S.A, Ammar, S.I. (2009). Transient solution of the M/M/c queue with balking and reneging. Computers and Mathematics with Applications, 57(8), 1280–1285.

[6] Ancker. Jr., C.J. and Gafarian A.V. (1963a) Some queuing problems with balking and reneging I. Operations Research, 11(1), 88–100.

[7] Ancker. Jr., C.J. and Gafarian A.V. (1963b) Some queuing problems with balking and reneging II. Operations Research, 11(6), 928–937.

[8] Boots, N. and Tijms, H. (1999). An M/M/c queue with impatient customers. TOP, 7(2), 213–220.

[9] Chao, X. (1995) A queuing network model with catastrophes and product form solution. Operations Research Letters, 18, 75–79.

[10] Dharmaraja, S. and Kumar, R. (2015). Transient solution of a Markovian queuing model with heterogeneous servers and catastrophes. Opsearch, 52(4), 810–826.

[11] DiCrescenzo, A., Giorno, V. and Nobile, A.G. (2003). On the M/M/1 queue with catastrophes and its continuous approximation. Queueing Systems, 43, 329–347.

[12] Di Crescenzo, A., Giorno, V., Nobile, A.G. and Ricciardi, L.M. (2009). A note on birthdeath processes with catastrophes. Statistics and Probability Letters, 78 (14), 2248–2257.

[13] Falin, G.I. and Artalejo, J.R. (1995). Approximations for multi-server queues with balking/retrial discipline. OR Spectrum, 17(4), 239–244.

[14] He, X. and Hu, W. (2014). Modeling Relief Demands in an Emergency Supply Chain System under Large-Scale Disasters Based on a Queuing Network. The Scientific World Journal, 2014. DOI: 10.1155/2014/195053.

[15] Haight, F.A. (1957). Queuing with balking. Biometrika, 44, 360–369.

[16] Haight, F.A. (1959). Queuing with reneging. Metrika, 2, 186–197.

[17] Ibrahim, R. and Whitt, W. (2009). Real-time delay estimation in overloaded multi-server queues with abandonments. Management Science, 55(10), 1729–1742.

[18] Janacek, J. and Kvet, M. (2017). An approach to uncertainty in emergency service systems via scenarios and fuzzy values. Croatian Operational Research Review, 8, 237–248.

[19] Jain, N. K. and Kumar, Rakesh (2007). Transient solution of a catastrophic-cum-restorative queuing problem with correlated arrivals and variable service capacity. International Journal of Information and Management Science, 18, 461–465.

[20] Jain, N. K. and Kanethia, D. K. (2006). Transient analysis of a queue with environmental and catastrophic effects. International Journal of Information and Management Science, 17(1), 35–45.

[21] Kumar, B. Krishna and Arivudainambi, D. (2000). Transient solution of an M/M/1 queue with catstrophes. Computers and Mathematics with applications, 40, 1233–1240.

[22] Kumar, B. Krishna, Madheshwari, S. Pavai and Venkatakrishanan, K. S. (2007). Transient solution of an $M/M/2$ queue with heterogeneous servers subject to catastrophes. International Journal of Information and Management Science, 18, 63–80.

[23] Kumar, B. K., Parthasarthy, P. R. and Sharafali, M. (1993). Transient solution of M/M/1 queue with balking. Queuing Systems, 13, 441–448.

[24] Kumar, B. K. and Pavai Madheswari, S. (2005). Transient Analysis of an M/M/1 Queue Subject to Catastrophes and Server Failures. Stochastic Analysis and Applications, 23, 329–340.

[25] Kumar, B. K., Krishnamoorthy, A., Pavai Madheswari, S. and Sadiq Basha, S. (2007). Transient analysis of a single server queue with catastrophes, failures and repairs. Queueing Systems, 56,133–141.

[26] Kumar, B. K., Vijayakumar, A. and Sophia, S. (2008). Transient Analysis for State-Dependent Queues with Catastrophes. Stochastic Analysis and Applications, 26, 1201–1217.

[27] Kumar, R. and Sharma, S.K. (2012a). An M/M/1/N queuing system with retention of reneged customers. Pakistan Journal of Statistics Operation Research, 8, 859–866.

[28] Kumar, R. and Sharma, S.K. (2012b). An M/M/1/N queuing model with retention of reneged customers and balking. American Journal of Operations Research, 2, 1–5.

[29] Montazer-Hagighi, A., Medhi, J. and Mohanty, S.G. (1986). On a multi-server Markovian queuing system with balking and reneging. Computers and Operations Research, 13(4), 421–425.

[30] Montazer Haghighi, Aliakbar (1998). An analysis of a parallel multi-processor system with task split and feedback. Computers and Operations Research, 25(11), 948–956.

[31] Obert, E.R. (1979). Reneging phenomenon of single channel queues. Mathematics of

Operations Research, 4, 162-178.

[32] Paz, N. and Yechiali, U. (2014). An M/M/1 queue in random environment with disasters. Asia-Pacific Journal of Operational Research, 31(3). DOI: 10.1142/S021759591450016X.

[33] Raju, S.N., Bhat, U.N. (1982). A computationally oriented analysis of the G/M/1 queue. Opsearch, 19:67–83.

[34] Robert J. Lopez, Maple via Calculus. Birkhuser, Boston, MA. DOI: 10.1007/978-1-4612-0267-7.

[35] Shin, Y.W. and Choo, T.S. (2009). M/M/s queue with impatient customers and retrials. Applied Mathematical Modelling, 33(6), 2596–2606.

[36] Subba Rao, S. (1965). Queueing models with balking, reneging and interruptions. Operations Research, 13, 596–608.

[37] Subba Rao, S. (1967). Queueing models with balking and reneging in M/G/1 system. Metrika, 2(1), 173–188.

[38] Sudhesh, R, Savitha P., Dharmaraja, S. (2016). Transient analysis of a two-heterogeneous servers queue with system disaster, server repair and customers impatience. TOP, DOI: 10.1007/s11750-016-0428-x.

[39] Sudhesh, R. (2010). Transient analysis of a queue with system disasters and customer impatience. Queueing Systems, 66(1), 95–105

[40] Tarabia, A. M. K. (2011). Transient and steady-state analysis of an $M/M/1$ queue with balking, catastrophes, server failures and repairs. Journal of Industrial and Management Optimisation, 7, 811–823.

[41] Vasiliadis, G. (2014). Transient analysis of the M/M/k/N/N queue using a continuous time homogeneous Markov system with finite state size capacity. Communications in Statistics-Theory and Methods, 43,1548–1562.

[42] Whitt, W. (1983). Untold horrors of the waiting room: what the equilibrium distribution will never tell about the queue length process. Management Science, 29, 395–408.

[43] Xiong, W. and Altiok, T. (2009). An approximation for multi-server queues with deterministic reneging times. Annals of Operations Research, 172(1), 143–151.

[44] Yechiali, Uri (2007). Queues with system disasters and impatient customers when system is down. Queuing Systems, 56(3):195–202.

[45] Zohar, E., Mandelbaum, A. and Shimkin, N. (2002). Adaptive behaviour of impatient customers in tele-queues: theory and empirical support. Management Science, 48(4), 566–583.