Non-linear averaging-based operators of pseudo-hesitant fuzzy elements and an application

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Abstract. Data modeling/aggregating, in many uncertain real-world' problems such as decision-making processes, has gotten more attention in recent years. Due to a variety of uncertainty sources, various types of fuzzy sets, and various types of averaging-based aggregation functions have been proposed. The power average operator (PAO), as a nonlinear operator, is more appropriate than other averaging-based functions for situations where different values are given on a single subject. In this paper, PAO will be extended to be used in the aggregation process of given pseudo-hesitant fuzzy elements (pseudo-HFEs), and some needed properties have been discussed, too. Then, four kinds of PAO with pseudo-HFEs, i.e., power average operator of pseudo-HFEs, power weighted average operator of pseudo-HFEs, power ordered weighted average operator of pseudo-HFEs and power hybrid average operator of pseudo-HFEs, will be defined. To solve a multi-attribute group decision-making (MAGDM) problem, the evaluation step done by both decision-makers and self-assessment will be quantified by pseudo-HFEs. Then the PAO will be applied to aggregate the row elements of the resulting decision matrix. The ranking orders of obtained pseudo-HFEs, show the options' orders. Finally, the proposed method will be used to solve a multi-attribute group decision-making problem, illustrated numerically, analyzed, and validated.

Keywords: aggregation function, multi attribute group decision making, power average, pseudo-hesitant fuzzy elements, score functions.

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1. Introduction

Data processing/mining, which are usually modeled numerically, are important tasks in many real-world problems, such as decision-making processes. Many theories [18, 22], pythagorean fuzzy sets [2], neutrosophic sets [24], hesitant fuzzy sets (HFSs) [6], hesitant fuzzy numbers (HFNs) [7], etc., have long been proposed to model uncertain information. Since the presentation of HFSs, they have absorbed many researchers and defined some other needed tools in their practical applications [19, 7]. It can be explained in terms of the threefold classification of real-world' problems, i.e., organized simplicity problems, organized complexity problems, and disorganized complexity problems [9].
Contrary to the previously mentioned triple categories of real-world problems, and diversity of uncertainty sources, HFSs have been extended to new kinds [10, 14, 1], recently. Also, HFSs have been extended to real values, which are called HFNs and generalized HFNs (GHFNs)[7, 8].

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As the HFNs in their current format may not be satisfied in normality condition, which is necessary for fuzzy numbers, we called them pseudo-HFEs in this paper.

Some practical decision-making problems involve a finite set of options that must be ranked against a finite set of criteria. Such issues are called multi-attribute decision making (MADM) problems, and there are many methods to solve them, especially in uncertain environments [12]. PAO as a non-linear averaging operator [21], was extended to aggregate vague/uncertain information and applied with HFSs to solve a group decision-making problem [11, 13, 16].

Making a good decision or finding a good solution to a MADM problem, clearly entails how to do well phases of evaluation, modeling, and aggregation, simultaneously. In this article, we will propose some methods based on self-assessment and the direct judgments of evaluators decision-makers (DMs), to reach this target.

I) With the aim of enjoying the benefits of impartial refereeing and self-evaluation of knowledgeable managers, they will be combined to achieve results closer to reality.

II) In the modeling phase, we'll use the pseudo-HFEs.

III) In the aggregation phase, due to the definition of support function based on pairwise distance measures of input pseudo-HFEs, new methods for arithmetic operations of pseudo-HFEs and their score function using PA operator will be suggested. Then, PA operator will be extended to aggregate pseudo-HFEs, power average operator of pseudo-HFEs ($PA-HFN$), power weighted average operator of pseudo-HFEs ($PWA-HFN$), power ordered weighted average operator of pseudo-HFEs ($POWA-HFN$) and power hybrid average operator of pseudo-HFEs ($PHA-HFN$) operators will be proposed.

This generalization of the PA operator on HFNs has a fundamental difference with the Yager' method in how to compute the total support $T(\tilde{A}_i^H)$. Also, we will prove that the $PA-HFN$ operator, unlike the $HFPA$ operator [23], is bounded and idempotent. The hybrid method will be applied to solve a MADM problem and the validity of the obtained result will be checked by criteria validity tests [17].

In the following: Some useful preliminaries and definitions that are necessary for other sections will be reviewed in Section 2. Section 3 presents the newly proposed score function and a new method for the arithmetic operation of pseudo-HFEs. The power average of pseudo-HFEs and their varieties, solving a MADM problem with pseudo-HFEs are expressed in Sections 4 and 5. The numerical description of the proposed methods and their validation will be presented in Section 6. The conclusion of this paper is given in Section 7.

2. Definitions and Preliminaries

In this section, some basic concepts which, are necessary for other parts will be reviewed.

Definition 1 [4] Let $\mathbb{I}$ be an interval from real numbers. An aggregation function in $\mathbb{I}^n$ is a non-decreasing and bounded function $A^{(n)}:\mathbb{I}^n \to \mathbb{I}$ that

$$inf_{\mathbf{x}\in\mathbb{I}^n} A^{(n)}(\mathbf{x}) = inf\mathbb{I}, \qquad sup_{\mathbf{x}\in\mathbb{I}^n} A^{(n)}(\mathbf{x}) = sup\mathbb{I},$$

and

$i)A^{(1)}(x)=x,$ $\qquad ii)A^{(n)}(x,x,...,x)=x,$

$iii)A^{(n)}(x_1,x_2,...,x_n)\le A^{(n)}(y_1,y_2,...,y_n)$ if $\forall 1\le i\le n$ we have $x_i\le y_i$.

For arbitrary values, $\mathfrak{a}$ and $\mathfrak{b}$, $Supp(\mathfrak{a},\mathfrak{b})$ is called the support for $\mathfrak{a}$ from $\mathfrak{b}$ and satisfied in:

1)$Supp(\mathfrak{a},\mathfrak{b})=Supp(\mathfrak{b},\mathfrak{a})$; $\qquad$ 2)$Supp(\mathfrak{a},\mathfrak{b})\in[0,1]$;

3) If $|x-y|<|\mathfrak{a}-\mathfrak{b}|$, then $Supp(\mathfrak{a},\mathfrak{b})<Supp(x,y)$.

Definition 2[16] PA of values $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$, is denoted by $PA(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n)$ and defined as

$$PA(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) = \frac{\sum_{i=1}^{n}(1 + T(\mathfrak{a}_i))\mathfrak{a}_i}{\sum_{i=1}^{n}(1 + T(\mathfrak{a}_i))},$$

where, $T(\mathfrak{a}_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Supp(\mathfrak{a}_i, \mathfrak{a}_j)$.

Definition 3[12] Let X be a fixed set. An HFS E on X is in terms of a function h that belongs to a subset of $[0, 1]$.

In general [12], $E = \{< x, h(x) > |x \in X\}$ is a mathematical representation of the defined HFS E on X, and hesitant fuzzy element (HFE) $h(x)$, as membership degrees of $x \in X$ to the set E, is a finite set of some values in $[0, 1]$.
Arithmetic operations and aggregation functions of HFEs are needed in some real applications [12].
Definition 4[12] Let $h_j (j = 1, 2, \dots, n)$ be a collection of HFEs, and $w = (w_1, w_2, \dots, w_n)$ with $w_i \in [0, 1], \sum_{i=1}^{n} w_i = 1$ be the weight vector of given HFEs, then
(i) An adjusted hesitant fuzzy weighted averaging $(AHFWA)$ operator is a mapping $F^n \to F$ such that $AHFWA(h_1, h_2, \dots, h_n) = \oplus_{j=1}^{n} w_j h_j = \left\{1 - \Pi_{j=1}^{n}(1 - h_j^{\sigma(t)})^{w_j} | t = 1, 2, \dots, l\right\}$,
(ii) An adjusted hesitant fuzzy weighted geometric $(AHFWG)$ operator is a mapping $F^n \to F$ such that $AHFWG(h_1, h_2, \dots, h_n) = \otimes_{j=1}^{n}(h_j)^{w_j} = \left\{\Pi_{j=1}^{n}(h_j^{\sigma(t)})^{w_j} | t = 1, 2, \dots, l\right\}$.

HFSs were also extended to a new type, called HFNs, which includes the real part and membership part [7], and is defined as follows.

Definition 5[7] Let X be the reference set and $a \in \mathbb{R}$, an HFN $\tilde{A}^H$ in the set of real numbers $\mathbb{R}$ is defined as $\tilde{A}^H = \langle a, h(a)\rangle$, where HFE $h(a)$ is a finite set of some values in $[0, 1]$, are considered as membership degrees of $a \in X$.

Definition 6[7] Let $\tilde{A}^H = \langle a, h(a)\rangle$ and $\tilde{B}^H = \langle b, h(b)\rangle$ be two HFNs and $\lambda > 0$, then
(1) $\tilde{A}^H \oplus \tilde{B}^H = \langle a + b, h(a) \cup h(b)\rangle$, where $h(a) \cup h(b) = \bigcup_{\gamma_1 \in h(a), \gamma_2 \in h(b)} \max\{\gamma_1, \gamma_2\}$
(2) $\lambda\tilde{A}^H = \langle \lambda a, h(a)\rangle$,
(3) $(\tilde{A}^H)^\lambda = \langle a^\lambda, h(a)\rangle$,
(4) $\tilde{A}^H \otimes \tilde{B}^H = \langle a.b, h(a) \cap h(b)\rangle$, where $h(a) \cap h(b) = \bigcup_{\gamma_1 \in h(a), \gamma_2 \in h(b)} \min\{\gamma_1, \gamma_2\}$.

Definition 7[7] Let $\tilde{A}^H = \langle a, h(a)\rangle$ with $h(a) = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be a HFN that $\gamma_i \in [0, 1]$ are possible satisfaction degrees. Then
(1) mean value (or the score function) of HFN $\tilde{A}^H$ is displayed as $MV(\tilde{A}^H)$ and defined as

$$MV(\tilde{A}^H) = \frac{a}{n}\sum_{i=1}^{n}\gamma_i.$$

(2) hesitant degree of HFN $\tilde{A}^H$ is defined as $\Pi(\tilde{A}^H) = a\sqrt{\frac{1}{n}\sum_{i=1}^{n}(MV(\tilde{A}^H) - \gamma_i)^2}$.

Definition 8[3] Consider two HFNs $\tilde{A}^H = \langle a, h(a)\rangle$ and $\tilde{B}^H = \langle b, h(b)\rangle$ where, $a, b \in \mathbb{R}$, $h(a)$ with cardinality $|h(a)| = k$ and $h(b)$ with cardinality $|h(b)| = l$ are two sets of some values in $[0, 1]$. The Euclidean distance (d_e) and Hamming distance (d_h) of these two HFNs were be defined as follows:

$$d_e(\tilde{A}^H, \tilde{B}^H) = \sqrt{\frac{1}{1+k\times l}\left(|a-b|^2 + \sum_{\substack{\gamma_i\in h(a),\\ \gamma_j\in h(b)}} |\gamma_i - \gamma_j|^2\right)}, \quad (1)$$

$$d_h(\tilde{A}^H, \tilde{B}^H) = \frac{1}{1+k\times l}\left(|a-b| + \sum_{\substack{\gamma_i\in h(a),\\ \gamma_j\in h(b)}} |\gamma_i - \gamma_j|\right). \quad (2)$$

3. Some new concepts about pseudo-HFEs

In this section, considering that in the common definition of HFNs, all degrees of hesitation may be strictly less than one, the normality condition is not fulfilled, we renamed the HFNs to pseudo-HFEs, and the Def. 5 will be corrected as Def. 9, firstly. Then, the arithmetic operations of pseudo-HFEs, and the score function of pseudo-HFEs will be updated.

Definition 9 Let X be the reference set. An extended HFS on X, with mathematical representation $\tilde{E} = \{\langle x, \tilde{H}(x)\rangle | x \in X\}$, is in terms of a function $\tilde{H}(x)$ that for each $x \in X$ returns a pseudo-HFEs $\tilde{A}^H = \langle a, h(a)\rangle$, in which $a \in \mathbb{R}$, and $h(a)$ is a finite set of some values in $[0, 1]$.

For simplicity the pseudo-HFEs $\tilde{A}^H = \langle a, h(a)\rangle$ will be used in practical applications of extended HFS $\tilde{E}$.

Definition 10 Let $\tilde{A}^H = \langle a, h(a)\rangle$ and $\tilde{B}^H = \langle b, h(b)\rangle$ be two pseudo-HFEs and $\lambda > 0$, then
(1) $\tilde{A}^H \oplus \tilde{B}^H = \langle a+b, h(a)\cup h(b)\rangle$, (2) $\lambda\tilde{A}^H = \langle \lambda a, h(a)\rangle$,
(3) $(\tilde{A}^H)^\lambda = \langle a^\lambda, h(a)\rangle$,
(4) $\tilde{A}^H \otimes \tilde{B}^H = \langle a.b, h(a)\cap h(b)\rangle$, if $h(a)\cap h(b) = \emptyset$, then $h(a)\cap h(b) = \bigcup_{\gamma_i\in h(a),\ \gamma_j\in h(b)} \min\{\gamma_i, \gamma_j\}$.

The new proposed score function is based on PA operator of pseudo-HFEs.

Definition 11 Let $\tilde{A}^H = \langle a, h(a) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\rangle$ be a pseudo-HFEs. Then,
(1) its score can be obtained as:

$$Score(\tilde{A}^H) = a \times PA(\gamma_1, \gamma_2, \ldots, \gamma_n) = a \times \frac{\sum_{j=1}^{n}(1+T(\gamma_j))\gamma_j}{\sum_{j=1}^{n}(1+T(\gamma_j))}, \quad (3)$$

where, $T(\gamma_j) = \sum_{k=1,k\neq j}^{n} Supp(\gamma_j, \gamma_k)$, $Supp(\gamma_j, \gamma_k) = 1 - |\gamma_j - \gamma_k|$.
(2) its variance is

$$Var(\tilde{A}^H) = a \times \sqrt{\frac{1}{n-1}\sum_{\substack{i=1,\\ i\neq j}}^{n}(\gamma_j - \gamma_i)^2}. \quad (4)$$

Definition 12 Let $\tilde{A}_1^H$ and $\tilde{A}_2^H$ be two pseudo-HFEs to be compared. Then
$(i)\tilde{A}_1^H \prec \tilde{A}_2^H$ $(\tilde{A}_1^H \succ \tilde{A}_2^H)$ if $Score(\tilde{A}_1^H) < Score(\tilde{A}_2^H)$ $(Score(\tilde{A}_1^H) > Score(\tilde{A}_2^H))$;
$(ii)\tilde{A}_1^H \prec \tilde{A}_2^H$ $(\tilde{A}_1^H \succ \tilde{A}_2^H)$ if $Score(\tilde{A}_1^H) = Score(\tilde{A}_2^H)$ & $Var(\tilde{A}_1^H) > Var(\tilde{A}_2^H)$

$(Var(\tilde{A}_1^H) < Var(\tilde{A}_2^H))$;

Note: For any two pseudo-HFEs $\tilde{A}_1^H$ and $\tilde{A}_2^H$ with the same real part and differences in at least one of the elements of their membership sets, we cannot say $\tilde{A}_1^H = \tilde{A}_2^H$ if $Score(\tilde{A}_1^H) = Score(\tilde{A}_2^H)$ & $Var(\tilde{A}_1^H) = Var(\tilde{A}_2^H)$. In this case, they are compared based on the maximum values (optimistic case) or minimum values (pessimistic case) in their membership sets. If these values are equal, by considering the values in the next ranks, the process will be done.

Theorem 1 Consider two pseudo-HFEs $\tilde{A}^H = \langle a, h(a)\rangle$ and $\tilde{B}^H = \langle b, h(b)\rangle$, and let $\lambda > 0$. If $\tilde{A}^H \prec \tilde{B}^H$ $(\tilde{A}^H \succ \tilde{B}^H)$, then $\lambda\tilde{A}^H \prec \lambda\tilde{B}^H$ $(\lambda\tilde{A}^H \succ \lambda\tilde{B}^H)$.

Proof. Let $\tilde{A}^H = \langle a, h(a)\rangle$ and $\lambda > 0$. We know that $\lambda\tilde{A}^H = \langle \lambda a, h(a)\rangle$. Then, it is obvious that $Score(\lambda\tilde{A}^H) = \lambda a \times PA(h(a)) = \lambda Score(\tilde{A}^H)$, and $Var(\lambda\tilde{A}^H) = \lambda Var(\tilde{A}^H)$.
Now, let $\tilde{A}^H \prec \tilde{B}^H$. Regarding to Def. 12, one of the following is true:

$(i) Score(\tilde{A}^H) < Score(\tilde{B}^H)$. Then as in Eq. (3) and Def. 12, we have

$$a \times PA(h(a)) < b \times PA(h(b)) \Rightarrow \lambda a \times PA(h(a)) < \lambda b \times PA(h(b)) \Rightarrow \lambda\tilde{A}^H \prec \lambda\tilde{B}^H.$$

$(ii) Score(\tilde{A}^H) = Score(\tilde{B}^H)$ & $Var(\tilde{A}^H) > Var(\tilde{B}^H)$. As in Def. 12, it is easy to see that $\lambda Var(\tilde{A}^H) > \lambda Var(\tilde{B}^H)$, and then $\lambda\tilde{A}^H \prec \lambda\tilde{B}^H$.

$(iii) Score(\tilde{A}^H) = Score(\tilde{B}^H)$ & $Var(\tilde{A}^H) = Var(\tilde{B}^H)$. The proof is clear. □

In the following example, the new score function is used to rank pseudo-HFEs compared with the previous one.

Example 1 Let $\tilde{A}_1^H = \langle 3; \{.1, .2, .6, .7\}\rangle$ and $\tilde{A}_2^H = \langle 3; \{.2, .3, .4, .7\}\rangle$ be two pseudo-HFEs. Based on the proposed method in this paper, $Score(\tilde{A}_1^H) = 1.2, Score(\tilde{A}_2^H) = 1.167$, and then $\tilde{A}_1^H \succ \tilde{A}_2^H$, while by using the proposed method in [7] $\tilde{A}_2^H \succ \tilde{A}_1^H$, because their scores are equal and $\tilde{A}_2^H$ has less variance than $\tilde{A}_1^H$. As we can see from the given pseudo-HFEs, $\tilde{A}_1^H$ has two big membership degrees, which makes the result of our method acceptable.

4. Aggregation of pseudo-HFEs using PA operator

A new aggregation method for pseudo-HFEs will be introduced in this section.

Definition 13 Let $\tilde{A}_i^H = \langle a_i, h(a_i)\rangle, i = 1, 2, \ldots, n$ $(n \geq 3)$ be pseudo-HFEs, then

$$PA - HFN(\tilde{A}_1^H, \tilde{A}_2^H, \ldots, \tilde{A}_n^H) = \frac{\sum_{i=1}^{n}[(1 + T(\tilde{A}_i^H))\tilde{A}_i^H]}{\sum_{i=1}^{n}(1 + T(\tilde{A}_i^H))},$$

is called PA of pseudo-HFEs (PA-HFN) where, $T(\tilde{A}_i^H) = \sum_{j=1, j\neq i}^{n} Supp(\tilde{A}_i^H, \tilde{A}_j^H)$ and for each i, j: $Supp(\tilde{A}_i^H, \tilde{A}_j^H)$ is the support for pseudo-HFE $\tilde{A}_i^H$ from pseudo-HFE $\tilde{A}_j^H$, satisfying in the following properties:

1) $Supp(\tilde{A}_i^H, \tilde{A}_j^H) = Supp(\tilde{A}_j^H, \tilde{A}_i^H)$;

2) $Supp(\tilde{A}_i^H, \tilde{A}_j^H) \in [0, 1]$;

3) If $d_h(\tilde{A}^H, \tilde{B}^H) < d_h(\tilde{C}^H, \tilde{D}^H)$ then $Supp(\tilde{A}^H, \tilde{B}^H) > Supp(\tilde{C}^H, \tilde{D}^H)$, where d_h is the Hamming distance, which can be replaced by d_e (Euclidean distance).

Theorem 2 For a collection $\tilde{A}_i^H = \langle a_i, h(a_i)\rangle, i = 1, 2, \ldots, n$ $(n \geq 3)$ of pseudo-HFEs, their aggregated value by PA-HFN operator, i.e., $PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H)$ is also a pseudo-HFE, and then

$$PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) = \left\langle \frac{\sum_{i=1}^{n}[(1+T(\tilde{A}_i^H))a_i]}{\sum_{i=1}^{n}(1+T(\tilde{A}_i^H))}, \bigcup_{i=1}^{n} h(a_i) \right\rangle.$$

Proof. Based on Definitions 10 and 13, it is easy to see that the proposed $PA-HFN$ operator has two distinct types of behavior: the real part of the input will be affected by the PA operator, while the membership degrees of the inputs will be affected by the union operator. Therefore, the output of this operator will have two real and membership parts. □

Theorem 3 Suppose $\tilde{A}_i^H, i = 1, 2, \dots, n$ $(n \geq 3)$ be pseudo-HFEs. If the support function is assumed to be constant for all values: $Supp(\tilde{A}_i^H, \tilde{A}_j^H) = k$ $(i \neq j)$, then $PA-HFN$ is arithmetic average, i.e., $PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) = \frac{\sum_{i=1}^{n} \tilde{A}_i^H}{n}$.

Proof. For each $i \neq j$ set $Supp(\tilde{A}_i^H, \tilde{A}_j^H) = k$. Then, $T(\tilde{A}_i^H) = (n-1)k$ and
$PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) = \frac{\sum_{i=1}^{n}(1+(n-1)k)\tilde{A}_i^H}{\sum_{i=1}^{n}(1+(n-1)k)} = \frac{\sum_{i=1}^{n}(1+(n-1)k)\tilde{A}_i^H}{n(1+(n-1)k)} = \frac{\sum_{i=1}^{n}\tilde{A}_i^H}{n}$. □

Boundedness, *Commutativity* and *Idempotency* properties, for $PA-HFN$ operator will be discussed in the following theorem.

Theorem 4 Suppose $\tilde{A}_i^H, i = 1, 2, \dots, n$ $(n \geq 3)$ be pseudo-HFEs, then

1) The boundary condition is true for $PA-HFN$ operator, i.e.

$$min(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) \leq PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) \leq max(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H);$$

2) Idempotency is verified for $PA-HFN$, i.e.

$$PA-HFN(\tilde{A}^H, \tilde{A}^H, \dots, \tilde{A}^H) = \tilde{A}^H;$$

3) For any permutation $\{\tilde{A}_{(1)}, \tilde{A}_{(2)}^H, \dots, \tilde{A}_{(n)}^H\}$ of $\{\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H\}$, we have

$$PA-HFN(\tilde{A}_{(1)}^H, \tilde{A}_{(2)}^H, \dots, \tilde{A}_{(n)}^H) = PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H).$$

Proof. 1) For simplicity, let $w_i = \frac{1+T(\tilde{A}_i^H)}{\sum_{i=1}^{n}(1+T(\tilde{A}_i^H))}$. Then, $\forall i = 1, 2, \cdots, n : 0 \leq w_i \leq 1$, $\sum_{i=1}^{n} w_i = 1$, and

$$PA-HFN(\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H) = \sum_{i=1}^{n} \frac{1+T(\tilde{A}_i^H)}{\sum_{i=1}^{n}(1+T(\tilde{A}_i^H))} \tilde{A}_i^H = \sum_{i=1}^{n} w_i \tilde{A}_i^H.$$

Define $\tilde{A}_{(1)}^H = min\{\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H\}$, and $\tilde{A}_{(n)}^H = max\{\tilde{A}_1^H, \tilde{A}_2^H, \dots, \tilde{A}_n^H\}$. Then, we'll have

$$\forall\ i = 1, 2, \cdots, n : \tilde{A}_{(1)}^H \leq \tilde{A}_i^H \leq \tilde{A}_{(n)}^H \Rightarrow w_i\tilde{A}_{(1)}^H \leq w_i\tilde{A}_i^H \leq w_i\tilde{A}_{(n)}^H \qquad \text{(Based on Theorem 1)} \Rightarrow$$

$$\sum_{i=1}^{n} w_i\tilde{A}_{(1)}^H \leq \sum_{i=1}^{n} w_i\tilde{A}_i^H \leq \sum_{i=1}^{n} w_i\tilde{A}_{(n)}^H \Rightarrow \tilde{A}_{(1)}^H \sum_{i=1}^{n} w_i \leq \sum_{i=1}^{n} w_i\tilde{A}_i^H \leq \tilde{A}_{(n)}^H \sum_{i=1}^{n} w_i \Rightarrow$$

$$\tilde{A}_{(1)}^H \leq \sum_{i=1}^{n} w_i\tilde{A}_i^H \leq \tilde{A}_{(n)}^H.$$

That is $PA-HFN$ is a bounded operator.

2) Let $\tilde{A}_i^H=\tilde{A}^H, i=1,2,\dots n$, and $n\geq 3$, i.e., the given pseudo-HFEs are all equal. Then, according to [21], the support function of each pairwise of them is constant, i.e.,

$$\forall i\neq j:\ Supp(\tilde{A}_i^H,\tilde{A}_j^H)=k\Rightarrow T(\tilde{A}_i^H)=\sum_{j=1,j\neq i}^{n} Supp(\tilde{A}_i^H,\tilde{A}_j^H)=\sum_{i=1}^{n}k=(n-1)k.$$

Thus, $PA-HFN(\tilde{A}_1^H,\tilde{A}_2^H,\dots,\tilde{A}_n^H)=$

$$\frac{\sum\limits_{i=1}^{n}(1+T(\tilde{A}_i^H))\tilde{A}_i^H}{\sum\limits_{i=1}^{n}(1+T(\tilde{A}_i^H))}=\frac{\sum\limits_{i=1}^{n}(1+(n-1)k)\tilde{A}^H}{n(1+(n-1)k)}=\frac{n(1+(n-1)k)\tilde{A}^H}{n(1+(n-1)k)}=\tilde{A}^H.$$

That is $PA-HFN$ is an idempotent operator.

3) Let $\{\tilde{A}_{(1)},\tilde{A}_{(2)}^H,\dots,\tilde{A}_{(n)}^H\}$ be any permutation of given pseudo-HFEs $\{\tilde{A}_1^H,\tilde{A}_2^H,\dots,\tilde{A}_n^H\}$. Then for each $1\leq j\leq n$ there exist one and only one i $(1\leq i\leq n)$ such that $\tilde{A}_{(j)}=\tilde{A}_i$. Therefore, $\forall 1\leq i,j\leq n$ we have $T(\tilde{A}_{(j)}^H)=T(\tilde{A}_i^H)$, and

$$PA-HFN(\tilde{A}_1^H,\tilde{A}_2^H,\dots,\tilde{A}_n^H)=\left\langle\frac{\sum\limits_{i=1}^{n}[(1+T(\tilde{A}_i^H))a_i]}{\sum\limits_{i=1}^{n}(1+T(\tilde{A}_i^H))},\bigcup_{i=1}^{n}h(a_i)\right\rangle=$$

$$\left\langle\frac{\sum\limits_{j=1}^{n}[(1+T(\tilde{A}_{(j)}^H))a_{(j)}]}{\sum\limits_{j=1}^{n}(1+T(\tilde{A}_{(j)}^H))},\bigcup_{j=1}^{n}h(a_{(j)})\right\rangle=PA-HFN(\tilde{A}_{(1)}^H,\tilde{A}_{(2)}^H,\dots,\tilde{A}_{(n)}^H).$$

□

The PA-HFN operator would be extended for pseudo-HFEs with different importance degrees. Then, power weighted average operator of pseudo-HFEs (PWA-HFN) can be defined.

Definition 14 Let $w_i\geq 0$ with $\sum\limits_{i=1}^{n}w_i=1$ be the weight of ith pseudo-HFEs $\tilde{A}_i^H, i=1,2,\dots,n$ $(n\geq 3)$, then

$$PWA-HFN(\tilde{A}_1^H,\tilde{A}_2^H,\dots,\tilde{A}_n^H)=\frac{\sum\limits_{i=1}^{n}[(1+w_iT(\tilde{A}_i^H))w_i\tilde{A}_i^H]}{\sum\limits_{i=1}^{n}(1+w_iT(\tilde{A}_i^H))w_i},\tag{5}$$

where, $T(\tilde{A}_i^H)=\sum\limits_{j=1,j\neq i}^{n}Supp(\tilde{A}_i^H,\tilde{A}_j^H)$.

Definition 15 Let for $i=1,2,\dots,n$ $(n\geq 3)$, $\tilde{A}_{(i)}^H$ be a permutation of pseudo-HFEs $\tilde{A}_i^H$, such that $\tilde{A}_{(1)}^H\leq\tilde{A}_{(2)}^H\leq\dots\leq\tilde{A}_{(n)}^H$, then $POWA-HFN(\tilde{A}_1^H,\tilde{A}_2^H,\dots,\tilde{A}_n^H)=\sum\limits_{k=1}^{n}\omega_k\tilde{A}_{(k)}^H$, where $\omega_k=\mathcal{Q}(\mathcal{R}_k/\mathcal{TV})-\mathcal{Q}(\mathcal{R}_{k-1}/\mathcal{TV}),\mathcal{R}_k=\sum\limits_{m=1}^{k}V(\tilde{A}_{(m)}^H),\mathcal{TV}=\sum\limits_{m=1}^{n}V(\tilde{A}_{(m)}^H),\ V(\tilde{A}_{(m)}^H)=1+T(\tilde{A}_{(m)}^H),\ \ \mathcal{Q}:[0,1]\longrightarrow[0,1]$ as a basic unit interval monotonic (BUM) function satisfied in: $\mathcal{Q}(0)=0,\mathcal{Q}(1)=1$, and for each $s>t$ then $\mathcal{Q}(s)>\mathcal{Q}(t)$. $Supp(\tilde{A}_{(m)}^H,\tilde{A}_{(l)}^H)$ is the support of lth largest value for mth largest value, and $T(\tilde{A}^H_{(m)})$ is the total support of mth largest pseudo-HFE by all the other ones, i.e., $T(\tilde{A}^H_{(m)}) = \sum_{l=1,\ l\neq m}^{n} Supp(\tilde{A}^H_{(l)}, \tilde{A}^H_{(m)})$.

Note: For some special forms of BUM function $\mathcal{Q}$, we get:

1) $\omega_k = V(\tilde{A}^H_{(k)})/\mathcal{TV} = [1 + T(\tilde{A}^H_{(k)})]/\sum_{j=1}^{n}[1 + T(\tilde{A}^H_{(j)})]$, and then, if $\mathcal{Q}(x) = x$

$$POWA - HFN(\tilde{A}^H_1, \tilde{A}^H_2, \ldots, \tilde{A}^H_n) = PA - HFN(\tilde{A}^H_1, \tilde{A}^H_2, \ldots, \tilde{A}^H_n).$$

2) If $\mathcal{Q}(x) = x$ and $Supp(\tilde{A}^H_{(l)}, \tilde{A}^H_{(j)}) = c, (c \in [0,1], l \neq j)$, we will have

$$\omega_k = V(\tilde{A}^H_{(k)})/\mathcal{TV} = [1 + T(\tilde{A}^H_{(k)})]/\sum_{j=1}^{n}[1 + T(\tilde{A}^H_{(j)})] = \frac{1}{n},$$

then POWA-HFN reduces to PA-HFN, i.e.,

$$POWA - HFN(\tilde{A}^H_1, \tilde{A}^H_2, \ldots, \tilde{A}^H_n) = \sum_{i=1}^{n} \tilde{A}^H_i/n.$$

3) POWA-HFN operator is called *max* or *min* operators if $\mathcal{Q}(x) = 1$ or $\mathcal{Q}(x) = 0$, respectively. Also, verifying boundary, commutativity and idempotent properties for POWA-HFN can be done easily.

When the given arguments are weighted, the POWA-HFN operator is extended to the power hybrid average (PHA) operator.

Definition 16 Let $W = (w_1, w_2, \ldots, w_n)$, with $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$ be the weight vector of pseudo-HFEs $\tilde{A}^H_i, i = 1, 2, \ldots, n$ $(n \geq 3)$. Then,

$$PHA - HFN_{\omega,w}(\tilde{A}^H_1, \tilde{A}^H_2, \ldots, \tilde{A}^H_n) = \sum_{k=1}^{n} \omega_k \tilde{A}^{H'}_{(k)},$$

is called the power hybrid average operator of pseudo-HFEs (PHA-HFN) where, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ satisfying in $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^{n} \omega_i = 1$ is associated vector, ρ in $\tilde{A}^{H'}_i = \rho w_i \tilde{A}^H_i$ $(i = 1, 2, \ldots, n)$ called balancing coefficient, and $\tilde{A}^{H'}_{(k)}$ is the kth largest of the weighted pseudo-HFE $\tilde{A}^{H'}_i (i = 1, 2, \ldots, n)$.

$T(\tilde{A}^H_i)$ which is called total support of $\tilde{A}^H_i$ by all the other pseudo-HFEs, will be obtained as in the following new algorithm.

4.1. A new algorithm for computing total support $T(\tilde{A}^H_i)$

Suppose $W = (w_1, w_2, \ldots, w_n)$ be the weight vector of n pseudo-HFEs $\tilde{A}^H_i, i = 1, 2, \ldots, n$ $(n \geq 3)$,

Step 1 By using Eqs. (1) or (2), compute Hamming (Euclidean) distance between of each pair of $\tilde{A}^H_i$ and $\tilde{A}^H_j, (i \neq j)$, are denoted by $d_{ij} = d_h(\tilde{A}^H_i, \tilde{A}^H_j)$ $(d_{ij} = d_e(\tilde{A}^H_i, \tilde{A}^H_j))$.

Step 2 Compute relative distance $rd_{ij} = \frac{d_{ij}}{\sum_{j=1, j\neq i}^{n} d_{ij}}$.

Step 3 Support for $\tilde{A}_i^H$ from $\tilde{A}_j^H$ is denoted by S_{ij} and obtain as $S_{ij} = Supp(\tilde{A}_i^H, \tilde{A}_j^H) = 1 - rd_{ij}$.

Step 4 Calculate the average support of $\tilde{A}_i^H$ (AS_i) by all other pseudo-HFEs, using the weighted power average operator as

$$AS_i = PWA(S_{i1}, \dots, S_{i,i-1}, S_{i,i+1}, \dots, S_{in}), \quad i = 1, 2, \dots, n.$$

Step 5 Normalize the average support as: $T(\tilde{A}_i^H) = \frac{AS_i}{\sum_{j=1}^{n} AS_j}$.

In the following example, we will numerically compare the performance of the introduced PA operators in [23] and in this paper.

Example 2 (Adapted from [23]) Consider four HFEs $a^1 = \{.8, .6\}, a^2 = \{.9, .5\}, a^3 = \{.7, .6\}$ and $a^4 = \{.7, .5\}$. Based on the defined hesitant fuzzy power average (HFPA) operator in [23], we have:

$$HFPA(a^1, a^1, a^1) = \{.8, .748, .6825, .6\},$$
$$HFPA(a^1, a^2, a^3) = \{.8175, .7992, .7692, .7459, .6910, .6599, .6090, .5697\},$$
$$HFPA(a^1, a^2, a^4) = \{.8183, .7846.7711, .7286, .6893, .6316, .6085, .5358\}.$$

As a result, the HFPA operator is neither idempotent nor bounded.
According to the definition of $PA - HFN$ operator, its input arguments must be pseudo-HFEs. So, in the first step we have to extend the HFEs into pseudo-HFEs. For instance, consider the following pseudo-HFEs: $\tilde{A}_H^1 = \langle 1; \{.8, .6\} \rangle, \tilde{A}_H^2 = \langle 1; \{.9, .5\} \rangle, \tilde{A}_H^3 = \langle 1; \{.7, .6\} \rangle$ and $\tilde{A}_H^4 = \langle 1; \{.7, .5\} \rangle$. Then

$$PA - HFN(\tilde{A}_H^1, \tilde{A}_H^1, \tilde{A}_H^1) = \langle 1; \{.8, .6\} \rangle = \tilde{A}_H^1,$$
$$PA - HFN(\tilde{A}_H^1, \tilde{A}_H^2, \tilde{A}_H^3) = \langle 1; \{.5, .6, .7, .8, .9\} = \tilde{B}_H^1,$$
$$PA - HFN(\tilde{A}_H^1, \tilde{A}_H^2, \tilde{A}_H^4) = \langle 1; \{.5, .6, .7, .8, .9\} = \tilde{B}_H^2.$$

Based on Eq. (3), Eq. (4) and Definition 12, we have $\tilde{A}_H^4 \prec \tilde{A}_H^3 \prec \tilde{A}_H^2 \prec \tilde{A}_H^1$, $\tilde{A}_H^3 \prec \tilde{B}_H^1 \prec \tilde{A}_H^1$ and $\tilde{A}_H^4 \prec \tilde{B}_H^2 \prec \tilde{A}_H^1$. Because, $Score(\tilde{A}_H^3) < Score(\tilde{B}_H^1) < Score(\tilde{A}_H^1)$, and $Score(\tilde{A}_H^4) < Score(\tilde{B}_H^2) < Score(\tilde{A}_H^1)$. Then, $HFN - PA$ is an idempotent and bounded operator.

5. PAO and solving MADM problems with pseudo-HFEs

Consider finite sets $A = \{A_1, A_2, \dots, A_m\}$ and $C = \{c_1, c_2, \dots, c_n\}$ as the sets of alternatives and criteria/attributes, respectively. Ranking of the alternatives can be done via the self-assessment method or direct-assessment method. Each candidate, in the self-assessment method, evaluates himself/herself based on pre-designed forms and gives himself/herself points. The scores obtained from the self-assessment method are analyzed by a group of experts, and the degree of its conformity with the actual performance of the candidate is expressed with a value between 0 and 1. Therefore, with each score we will have a finite set of values between 0 and 1, which is called the pseudo-HFE $\langle a_{ij}, \{\gamma_1, \dots, \gamma_k\} \rangle$, i.e., the crisp score a_{ij} of ith candidate with respect to jth criterion has been questioned by a group of k experts with a degree of $\gamma_i \in [0, 1]$. However, in the direct-assessment method, each of the options is evaluated by the decision-maker about each of the criteria, which is expressed by pseudo-HFEs $\tilde{h}_{ij} = \langle a_{ij}, \{\gamma_1, \dots, \gamma_k\} \rangle$, $\gamma_i \in [0, 1]$ is called hesitation degree of DM about his/her assessment. In both cases, the scores of the options are arranged in a matrix called the hesitant decision matrix $H\tilde{F}ND = [\tilde{h}_{ij}] = [\langle a_{ij}, \{\gamma_1, \dots, \gamma_k\} \rangle]_{m \times n}$.

Let $W = (w_1, w_2, \ldots, w_n)$ be the weight vector of criteria. Such a MADM problem can be solved using PA operators of pseudo-HFE as follows:

Step 1: Pick ith row $(i = 1, 2, \ldots, m)$ of the hesitant decision matrix $H\tilde{FND} = [\tilde{h}_{ij}]$, and based on the proposed algorithm in Subsection 4.1, compute the total support $T(\tilde{h}_{ij})$ $(j = 1, 2, \ldots, n)$.

Step 2: Aggregate each row of hesitant decision matrix $H\tilde{FND}$ using $PWA - HFN$ operator as in Eq. (5), and the obtained values in Step 1.

Step 3: Compute and rank the score function and variance of the obtained pseudo-HFEs in Step 2.

Step 4: Reorder the options $A_i, i = 1, 2, \ldots, n$, according to the ranking order of their pseudo-HFEs points in Step 3.

6. Numerical example

Example 3 Consider a student who is participated in courses Math, Art, and Physics, which are weighted by 0.4, 0.3 and 0.3, respectively. Suppose that the professors of each course are allowed to have a final exam and an arbitrary number of qualitative evaluations of the student's scientific activities during the semester. Let, $\langle 87; \{0.9, 0.7, 0.9, 0.9, 0.9\}\rangle$, $\langle 95; \{0.8, 0.9, 0.95, 0.9, 0.9\}\rangle$, and $\langle 90; \{0.7, 0.75, 0.8, 0.85, 0.9\}\rangle$ be the student's assessment values in courses, in which the real part of each given pseudo-HFE is the student' gained point in the corresponding final exam. Then, using the PWA-HFN the student's average can be obtained as $\langle 90.25; \{0.7, 0.7, 0.75, 0.8, 0.8, 0.85, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.95\}\rangle$, with score 76.893.
The point is that in calculating the grade point average, in addition to the obtained grades, attention is paid to the dispersion and, in fact, the student's scientific fluctuations through the support function. This careful consideration prevents the large effect of temporary activities on the final output, in addition to maximizing effort on the night of the exam, students are encouraged to improve their quality of knowledge during the course.

6.1. Sensitivity analysis

Now, let's discuss some special cases and their effects in final results.
Case 1. *A student with minimal classroom activity but very successful in final exams*
Let the scores are given as $\langle 97; \{0.2, 0.2, 0.3, 0.3, 0.35\}\rangle$, $\langle 99; \{0.1, 0.1, 0.15, 0.2, 0.2\}\rangle$, and $\langle 98; \{0.1, 0.15, 0.2, 0.25, 0.3\}\rangle$. The average point of this student is $\langle 97.93; \{0.1, 0.15, 0.2, 0.3, 0.25, 0.35\}\rangle$, that its score is 20.174. This shows that although the student was able to achieve high success in the short period of exams, but the lack of proper classroom activity negatively affects this success.
Case 2. *A student with normal classroom activity and successful in final exams*
Let the scores are given as $\langle 78; \{0.5, 0.6, 0.65, 0.7, 0.75\}\rangle$, $\langle 82; \{0.6, 0.65, 0.7, 0.45, 0.8\}\rangle$, and $\langle 75; \{0.6, 0.65, 0.5, 0.55, 0.7\}\rangle$. In this case, $\langle 78.30; \{0.5, 0.55, 0.45, 0.6, 0.65, 0.7, 0.75, 0.8\}\rangle$, is the average point of student that its score is 48.94. Compared to the previous case, it is observed that the student has a higher final score, and this is logical. Because the student who is involved in the scientific topics of the class during the semester has a better academic quality than the student who has prepared for the exam only for a short time.

The use of pseudo-HFEs proved that final decisions, such as student rankings and the like, are not limited to showcase activities and activities limited to the assessment period. Also, their aggregation by the power average operator showed that large fluctuations in performance reduced the final assessment value. Therefore, students/ services providers will have enough motivation to strive continuously and maintain a high level of performance over time.

Example 4 (Evaluate organizations) (Adapted from [7]) Consider a MADM problem including organizations $\mathcal{O}_i$ $(i = 1, 2, \ldots, 7)$, and attributes a_i $(i = 1, 2, \ldots, 6)$ with the score ceiling $125, 330, 175, 90, 150$, and 130, respectively. In the first step, self-assessment forms are distributed among the organizations to evaluate themselves. Note, $\mathcal{O}_5$ is a virtual organization with unrealistic self-assessment points. Due to the large difference between the criteria scores ceiling, we first calculate the relative obtained score of each option. In the second stage, some experts (for example five experts) are invited to evaluate organizations, and expressed their opinions in this regard with values of the range of 0 and 1. Then, the obtained values are HFEs. In the third stage, these two disjoint assessment values are merged which results pseudo-hesitant fuzzy elements decision matrix $H\tilde{FN}D = [\langle h_{ij}, \{\gamma_1, \ldots, \gamma_5\}\rangle]_{7\times 6}$ with pseudo-HFEs elements, as follows:

$$H\tilde{FN}D = \begin{pmatrix}
\langle .992; \{0.3, 0.4, 0.5, 0.5, 0.2\}\rangle & \langle .973; \{0.1, 0.4, 0.7, 0.8, 0.9\}\rangle & \langle 1.00; \{0.2, 0.6, 0.6, 0.4, 0.5\}\rangle & \langle .911; \{0.7, 0.7, 0.5, 0.6, 0.9\}\rangle & \langle .960; \{0.3, 0.2, 0.6, 0.3, 0.3\}\rangle & \langle .978; \{0.3, 0.4, 0.6, 0.7, 0.7\}\rangle \\
\langle .984; \{0.3, 0.5, 0.8, 0.6, 0.9\}\rangle & \langle .982; \{0.3, 0.5, 0.6, 0.5, 0.9\}\rangle & \langle .914; \{0.9, 0.9, 0.9, 0.9, 0.9\}\rangle & \langle .978; \{0.7, 0.8, 0.5, 0.5, 0.9\}\rangle & \langle .980; \{0.3, 0.4, 0.4, 0.6, 0.8\}\rangle & \langle .961; \{0.8, 0.8, 0.8, 0.9, 0.9\}\rangle \\
\langle .984; \{0.3, 0.5, 0.6, 0.7, 0.9\}\rangle & \langle .991; \{0.1, 0.5, 0.6, 0.9, 0.9\}\rangle & \langle .971; \{0.3, 0.5, 0.7, 0.6, 0.9\}\rangle & \langle .944; \{0.7, 0.6, 0.5, 0.6, 0.9\}\rangle & \langle .987; \{0.3, 0.3, 0.5, 0.6, 0.6\}\rangle & \langle .977; \{0.6, 0.7, 0.8, 0.6, 0.9\}\rangle \\
\langle .968; \{0.9, 0.7, 0.8, 0.9, 0.9\}\rangle & \langle .997; \{0.1, 0.7, 0.3, 0.8, 0.9\}\rangle & \langle .989; \{0.2, 0.6, 0.7, 0.4, 0.5\}\rangle & \langle .944; \{0.8, 0.7, 0.5, 0.5, 0.6\}\rangle & \langle .980; \{0.1, 0.5, 0.4, 0.5, 0.9\}\rangle & \langle .992; \{0.7, 0.3, 0.3, 0.4, 0.5\}\rangle \\
\langle 1.00; \{0.1, 0.1, 0.1, 0.1, 0.1\}\rangle & \langle .991; \{0.1, 0.2, 0.1, 0.2, 0.1\}\rangle & \langle .977; \{0.2, 0.2, 0.3, 0.1, 0.1\}\rangle & \langle 1.00; \{0.1, 0.2, 0.2, 0.2, 0.2\}\rangle & \langle .993; \{0.1, 0.2, 0.1, 0.3, 0.3\}\rangle & \langle 1.00; \{0.3, 0.1, 0.2, 0.2, 0.1\}\rangle \\
\langle 1.00; \{0.8, 0.8, 0.9, 0.8, 0.9\}\rangle & \langle .985; \{0.9, 0.8, 0.7, 0.8, 0.9\}\rangle & \langle .983; \{0.2, 0.2, 0.3, 0.4, 0.5\}\rangle & \langle 1.00; \{0.3, 0.4, 0.5, 0.5, 0.2\}\rangle & \langle .967; \{0.1, 0.7, 0.6, 0.8, 0.9\}\rangle & \langle .992; \{0.5, 0.3, 0.7, 0.4, 0.5\}\rangle \\
\langle .976; \{0.4, 0.4, 0.5, 0.6, 0.9\}\rangle & \langle .985; \{0.8, 0.5, 0.6, 0.9, 0.6\}\rangle & \langle .983; \{0.3, 0.5, 0.6, 0.6, 0.6\}\rangle & \langle .989; \{0.3, 0.5, 0.5, 0.6, 0.9\}\rangle & \langle .980; \{0.7, 0.7, 0.5, 0.6, 0.8\}\rangle & \langle .992; \{0.3, 0.5, 0.6, 0.4, 0.4\}\rangle
\end{pmatrix}.$$

Let $W = (0.125, 0.330, 0.175, 0.09, 0.15, 0.13)$ be the weight vector of criteria. Ranking of the organizations can be done by the implementing of steps of the proposed algorithm.

Table 1: The total points of alternatives and their ranking orders

Organizations	Aggregated pseudo-HFEs	Score function	Ranking order
$\mathcal{O}_1$	$\langle 0.971, \{.1, .2, .3, .4, .5, .6, .7, .8, .9\}\rangle$	0.4855	6
$\mathcal{O}_2$	$\langle 0.967, \{.3, .4, .5, .6, .7, .8, .9\}\rangle$	0.5802	2
$\mathcal{O}_3$	$\langle 0.978, \{.1, .3, .5, .6, .7, .8, .9\}\rangle$	0.5602	3
$\mathcal{O}_4$	$\langle 0.981, \{.1, .2, .3, .4, .5, .6, .7, .8, .9\}\rangle$	0.4905	5
$\mathcal{O}_5$	$\langle 0.993, \{.1, .2, .3\}\rangle$	0.1986	7
$\mathcal{O}_6$	$\langle 0.987, \{.2, .3, .4, .5, .6, .7, .8, .9\}\rangle$	0.5429	4
$\mathcal{O}_7$	$\langle 0.984, \{.3, .4, .5, .6, .7, .8, .9\}\rangle$	0.5904	1

Step 1. By applying Eq. (5) on each row of the decision matrix $H\tilde{FN}D$, we get the second column of Table 1.
Step 2. By applying Eq. (3) on the obtained pseudo-HFEs in Step 1, their score functions can be obtained as in the third column of Table 1.
Step 3. By using the ranking order of the score functions from Step 2, $\mathcal{O}_7$ is the best organization, and we have $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_6 \succ \mathcal{O}_4 \succ \mathcal{O}_1 \succ \mathcal{O}_5$.

6.2. Validity test

In this subsection, the validity of the proposed method is examined through three test criteria, that Wang and Triantaphyllou [17] have proposed for checking and demonstrating the feasibility of MCDM methods.
Test criterion 1 If we replace a non-optimized option with a worse one while other conditions such as the weight vector remain constant, the position of the best option should not be changed.
Test criterion 2 In an effective MCDM method, the transitivity property must be established.

Test criterion 3 By using an effective MCDM method, we get similar rankings if the given MCDM problem:
i) be solved without breaking it down into several MCDM sub-problems,
ii) be decomposed into several MCDM sub-problems, based on its options set, firstly. Then, the new MCDM problems are solved, and the obtained rankings are merged into a total ranking.
Consider the organization $\mathcal{O}_3$ is replaced with a non-optimized option $\mathcal{O}$ which is evaluated as

$$\mathcal{O} = \big\{\langle 122; \{0.4, 0.7, 0.2, 0.5, 0.5\}\rangle, \langle 324; \{0.5, 0.2, 0.3, 0.6, 0.3\}\rangle, \langle 169; \{0.3, 0.8, 0.6, 0.6, 0.5\}\rangle, \\ \langle 87; \{0.4, 0.9, 0.1, 0.2, 0.2\}\rangle, \langle 145; \{0.6, 0.6, 0.5, 0.3, 0.3\}\rangle, \langle 126; \{0.4, 0.7, 0.6, 0.7, 0.7\}\rangle\big\}.$$

These partial assessment values are aggregated using PWA-HFN operator and then we have $PWA - HFN(\mathcal{O}) = \langle 0.974; \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ with score function 0.487. Based on it, we have $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_6 \succ \mathcal{O}_4 \succ \mathcal{O} \succ \mathcal{O}_1 \succ \mathcal{O}_5$. It shows the rank of the best organization is unchanged. Moreover, it is obvious that the transitivity property is satisfied, because the ranking order is obtained based on real values which are called score functions. Now, let we break the problem down into four not necessarily separate sub-problems $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7\}$, $\{\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_7\}$, $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_7\}$ and $\{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6\}$.
These new MADM sub-problems are solved using the PWA-HFN operator method, proposed in this paper, then we get $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_6 \succ \mathcal{O}_1 \succ \mathcal{O}_5$, $\mathcal{O}_7 \succ \mathcal{O}_3 \succ \mathcal{O}_4 \succ \mathcal{O}_5$, $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_4 \succ \mathcal{O}_1 \succ \mathcal{O}_5$ and $\mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_6 \succ \mathcal{O}_4 \succ \mathcal{O}_5$, respectively. Overall ranking order of organizations will be achieved through the combining these orders as $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_6 \succ \mathcal{O}_4 \succ \mathcal{O}_1 \succ \mathcal{O}_5$ which, is the same as the original ranking order.
Hence, the proposed method is valid under test criteria 1, 2 and 3.

6.3. Comparative analysis

Let us forget for a moments the concept of pseudo-HFEs, and solve Example 4 through two evaluation methods, i.e., using real parts and membership parts of elements in decision matrices $\tilde{HFND}$, separately. With the self-evaluation (real part of pseudo-HFEs), the direct sum gives $\mathcal{O}_5 \succ \mathcal{O}_6 \succ \mathcal{O}_7 = \mathcal{O}_4 \succ \mathcal{O}_3 \succ \mathcal{O}_1 \succ \mathcal{O}_2$. The simple additive weighting (SAW) operator ordered them as $\mathcal{O}_5 \succ \mathcal{O}_4 \succ \mathcal{O}_6 \succ \mathcal{O}_7 \succ \mathcal{O}_3 \succ \mathcal{O}_1 \succ \mathcal{O}_2$. Also, by TOPSIS method we will get $\mathcal{O}_5 \succ \mathcal{O}_3 \succ \mathcal{O}_4 \succ \mathcal{O}_6 \succ \mathcal{O}_7 \succ \mathcal{O}_1 \succ \mathcal{O}_2$. It can be seen that, in all these three methods, the best option is the virtual option $\mathcal{O}_5$, with unrealistic evaluation values. With the DMs' evaluation (membership part of the given pseudo-HFEs in decision matrices $\tilde{HFND}$), by applying the TOPSIS-CI method [3], we have: $\mathcal{O}_2 \succ \mathcal{O}_6 \succ \mathcal{O}_3 \succ \mathcal{O}_7 \succ \mathcal{O}_4 \succ \mathcal{O}_1 \succ \mathcal{O}_5$.

The completely opposite results of the above two individual cases show that relying on each lead to incorrect ranking. The combination of self-evaluation and DMs' evaluation, as a logical way, leads to the hesitant decision matrix $\tilde{HFND}$. This combination enables us to have more fair evaluations. Because on the one hand, unwanted tendencies are managed in the self-evaluation process, and on the other hand, the effects of non-expert judges are moderated. However, not paying attention to the dispersion of judges' opinions, i.e., the hesitant part of pseudo-HFEs in matrix $\tilde{HFND}$, as seen in the TOPSIS method, causes the final ranking to be far from reality: $\mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_4 \succ \mathcal{O}_6 \succ \mathcal{O}_7 \succ \mathcal{O}_1 \succ \mathcal{O}_5$. The proposed method of this article, i.e., the $PWA - HFN$ method, has the advantage of a positive reaction to the distance and proximity of evaluators' opinions. The effect of this reaction can be seen in the ranking of options in Example 4: $\mathcal{O}_7 \succ \mathcal{O}_2 \succ \mathcal{O}_3 \succ \mathcal{O}_6 \succ \mathcal{O}_4 \succ \mathcal{O}_1 \succ \mathcal{O}_5$.
The use of the combined matrix caused the virtual option $\mathcal{O}_5$ with unrealistic self-evaluation scores to be placed at the bottom of the rankings in both TOPSIS and $PWA - HFN$ methods. In this case, the ranking of the $PWA - HFN$ method is even more accurate than the TOPSIS method. To prove this claim, we compare the scores of the best options of these two methods,

that is, $\mathcal{O}_2$ and $\mathcal{O}_7$ options, according to each criterion:
i) From the point of view of the real part of the elements in the decision matrix $\tilde{HFND}$, option $\mathcal{O}_7$, except for the first criterion, has higher self-evaluation points, with less variance, than option $\mathcal{O}_2$ against other criteria.
ii) By comparing the hesitant part of the scores of these two options in the matrix $\tilde{HFND}$, although it seems that the second option has a better performance in most criteria from the evaluators, but the dispersions of opinions are such that considering the effect of variance, this advantage is neutralized.
As a result, the ranking obtained from the TOPSIS method has been slightly adjusted in the $PWA - HFN$ method. This shows that using the method proposed in this paper can produce better results when pseudo-HFEs are used.

7. Conclusion

To take advantage of pseudo-HFEs, methods for calculating their score function and their arithmetic operations are presented in this article. Then, some aggregation operators of pseudo-HFEs based on PAO, i.e. $PA - HFN, PWA - HFN, POWA - HFN$ and $PHA - HFN$ operators, have been proposed. Finally, a hybrid technique based on combining two evaluation methods: self-assessment and DMs' evaluations, has been proposed to solve MAGDM problems using pseudo-HFEs and PA operators. The advantage of the proposed method is that while having the strengths of these two methods, it will correct some wrong details, such as unrealistic scores due to low cognition of DMs or individual motivations of those evaluated.

Due to the novelty of the subject of pseudo-hesitant fuzzy elements [7], many methods for solving decision problems and mathematical concepts to use them will have to be updated in the future. Their application in solving linear programming problems, data envelopment analysis, inaccurate graphs, quantum decision-making problems [5, 15], etc., are other interesting future research topics.

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