On the American style futures contracts

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Abstract. There is a large number of sources devoted to the American style options. On the other hand, the American futures contracts are understudied in the scientific literature. This motivated us to examine these instruments in comparison to the relevant options. Their optimal boundaries are obtained and a finite difference scheme is applied to the pricing problem. We consider separately the long and short positions.

Keywords: American futures contracts, American options, optimal boundaries, pricing

Received: October 3, 2023; accepted: December 19, 2023; available online: May 27, 2024x

DOI: 10.17535/crorr.2024.0004

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1. Introduction

Financial derivatives are the basic instruments which the investors use against the financial risks. They are mainly two kinds – European and American. The difference between them appears in the expiration dates – the European style derivatives mature at a predefined date, whereas the holder of an American one has the right to choose this date. The most important instruments are the futures and options. A futures contract (long or short positioned) gives to its holder the right and the obligation to bye or sell some asset on a predefined delivery price. On the other hand, the options (call or put) provide the right without obligation to buy or sell the underling asset for the pre agreed price (the strike price). Thus, the call options are related to the long positioned futures, whereas the put ones are related to the short futures.

The oldest option exchange is the Philadelphia Stock Exchange founded in 1790 (NASDAQ OMX PHLX, https://www.nasdaq.com/solutions/nasdaq-phlx). However, the largest option market is the Chicago Board Options Exchange (CBOE, https://www.cboe.com/). We have to mention also the Frankfurt-based Eurex Exchange (https://www.eurex.com/ex-en/), Tokyo Stock Exchange (TSE), Taiwan Futures Exchange (TAIFEX), and two electronic platforms – Boston Options Exchange (BOX) and Miami International Securities Exchange (MIAX). Of course, futures contracts are traded at some of these financial markets too – for example Eurex Exchange, TAIFEX, etc. But the largest futures market is formed by the CME group (https://www.cmegroup.com/) which includes Chicago Mercantile Exchange (CME), Chicago Board of Trade (CBOT), New York Mercantile Exchange (NYMEX), Commodity Exchange Inc. (COMEX), Kansas City Board of Trade (KCBT), and the NEX Group. An electronic platform for futures trading is the Intercontinental Exchange (ICE, https://www.theice.com/index).

In fact, the American style futures are relatively rarely traded. Nevertheless, these instruments have its theoretical significance as well as its practical importance. On the other hand, the majority of the options traded are of the American style. All these explain the large number

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of sources for both style options, while the literature devoted to the American style futures is very limited. The purpose of this paper is to fill this gap. We do this comparing the American style futures with the related options – puts/calls with short/long futures.

We work in this paper under the Black-Scholes framework [3] assuming that the underlying asset follows a log-normal process. As a consequence, a European style derivative, particularly an option or a futures, can be evaluated through a partial differential equation (the Black-Scholes equation) related to the infinitesimal generator of the underlying stochastic process and terminal conditions specified by the particular financial instrument. On the contrary, pricing of American style derivatives is in fact an optimal decision problem due to the holder's right to chose the expiration date. It is well known that such tasks lead to the so-called free boundary problems – we have to find the solution as well as the region in which it holds. This set is known as the continuation region, whereas its complement is named an early exercise or optimal region. If the combination between the time to maturity and the underlying asset price belongs to the optimal set, then the immediate exercise is the best holder's strategy. The boundary between both sets is known as an early exercise boundary. In such a way, if the initial asset price is in the continuation region, then the holder's optimal strategy is the first hit to this boundary. The form of the regions divides the American style derivatives into two types. The first one leads to a connected optimal set – usual options, [25, 28], capped options, [5, 29], power options, [20], variable annuities, [13]. This way we have a first hitting problem to a line. The other class of American derivatives leads to an optimal region consisting of two parts – thus a first exit from a strip arises. Such instruments are the straddle options, [1, 9], their extensions named strangles, [6, 12, 14, 15, 21, 22, 32], cancelable American options, also known as Israeli or game options, [8, 16, 18, 24, 31, 30], etc.

The results of this paper can be summarized in several directions. As we can expect, the American futures contracts fall in the first class – all points above (below) the exercise boundary are optimal for the long (short) positions. Also, the dividend rate has a key role. If it is unavailable, then the American futures degenerate – the immediate exercise is always optimal for the short positions and never for the long ones. On the contrary, the dividends make the futures non-trivial instruments. It turns out that the optimal boundary of a perpetual instrument (note that it is flat) coincides with the boundary of the related option except when there are no dividends for a short positioned futures. This holds for the initial boundary values too for some particular risk-free and dividend rates. Generally, the put option leads to a lower boundary than a short futures. The opposite is true for the call options and the long futures.

Once we characterize the shape of the optimal boundary, we approximate it by exponent of piecewise linear functions. We do this using some first hitting properties of the Brownian motion to a piecewise linear function and maximizing the holder's financial utility at every step. This way we can view the American style derivative as a European one – it expires either at the maturity or when the underlying asset hits the already approximated boundary. Thus, the free boundary equation, which the American futures price solves, turns to a boundary value problem in a known region. We apply the Crank-Nicolson finite difference approach to approximate the fair price. We choose this method despite some possible oscillations it generates. They appear because the Crank-Nicolson method is A-stable but not L-stable. Nonetheless, we use it since it is unconditionally stable and fast enough. In addition, the futures payoff functions are differentiable unlike the option payoffs. For some alternative numerical methods and for a discussion on the arising differences we refer to [19]. Let us mention in addition the integral approach of [17] applied to American option pricing. It is based on solving several integral equations for approximating the optimal boundaries. This method can be also adapted to the futures contracts. However, numerical solving of these integral equations needs relatively much computation time. Also, the approach based on maximizing the holder's profit could be more preferable for the investors due to its intuitiveness.

The paper is organized as follows. We present the base we use later in Section 2. The optimal boundaries are discussed in Section 3 and the finite difference pricing method is provided in Section 4. Some numerical examples can be found in Section 5.

2. Preliminaries

Let the underlying asset be modeled by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t \tag{1}$$

under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. The risk-free rate is the constant r and thus the measure Q is risk-neutral. We shall follow [23] to present the dividends – let λ be an additional discount rate such that $\lambda \geq 0$ and $r + \lambda > 0$. It can be viewed as a dividend rate – proposition 2.3 from [26] states that a model with a dividend rate δ is equivalent to a non-dividend model with parameters $\overline{r} = r - \delta$ and $\overline{\lambda} = \lambda + \delta$. Note that we do not restrict r to be positive. Let K be the delivery price. Thus the payment function of the long futures can be written as $N(t, x) = e^{-\lambda t}(x - K)$, i.e. the holder receives amount of $N(t, x)$ if he/she exercises in a moment t at a price x. The short positions lead to the opposite function. The inequality

$$E^x\left[e^{-r\zeta}(S_\zeta - K)\right] = x - KE^x\left[e^{-r\zeta}\right] > x - K \tag{2}$$

written for an arbitrary finite strategy ζ confirms that the immediate exercise is always/never optimal for the short/long positions when $\lambda = 0$. We assume hereafter that $\lambda > 0$. We shall use the notation $F(t, x)$ for the price.

Remark 1. *The above-mentioned transformation shows that the (r, λ, δ)-model (a model with risk free rate r, additional discount factor λ, and dividend rate δ) is equivalent to a $(r + \lambda, 0, \delta + \lambda)$-model. Particularly, the models with parameters $(r, \lambda, 0)$ and $(r + \lambda, 0, \lambda)$ are equivalent. Hence, the imposed inequalities $\lambda > 0$ and $r + \lambda > 0$ define the model under the traditional assumptions of positive risk free and dividend rates. We shall work w.r.t. these restrictions despite that negative rates are not so unrealistic. The option pricing problem under negative levels is studied in [2, 10]. Note that some values lead to a non-connected continuation region and a double optimal boundary. We leave for further investigation the futures contracts under negative interest rates. Let us mention that both the optimal and continuation regions have to be connected. Also, the following inequalities hold for an arbitrary finite stopping time ζ and when $\lambda < 0$ and $r + \lambda > 0$*

$$\begin{aligned} &E^x\left[e^{-(r+\lambda)\zeta} S_\zeta\right] > x \\ &E^x\left[e^{-(r+\lambda)\zeta}\right] < 1. \end{aligned} \tag{3}$$

The opposite inequalities hold when $\lambda > 0$ and $r + \lambda < 0$. This means that the American futures degenerate when the rates λ and $r + \lambda$ have different signs.

3. Optimal boundaries

We shall examine and approximate the optimal boundary – we denote it by $c(\tau)$ w.r.t. the time to maturity τ. Similar arguments to [11] show that it is a decreasing function for the short futures. See also, [17] and [7]. Thus, all points below the boundary are optimal, whereas keeping the futures is a better strategy for the ones above the boundary. On the contrary, the long futures boundary increases and the optimal points are above it.

3.1. Initial values

The following proposition determines the value of the boundary when the time to maturity tends to zero.

Proposition 1. *We have $c(0) = \frac{r+\lambda}{\lambda} K$ for both the short and long futures.*

Proof. Let us consider a long position, the result for the short futures can be proven analogously. Suppose that a point x is optimal. Therefore, if we denote by $(\mathcal{A}f)$ the infinitesimal generator of process (1), then the variational inequality

$$\begin{aligned} 0 &> N_t(t,x) + \mathcal{A}N(t,x) - rN(t,x) \\ &= (r+\lambda)K - \lambda x \end{aligned} \tag{4}$$

must be true. Hence, $c(0) \geq \frac{r+\lambda}{\lambda} K$. It remains to be proven that the equality holds. Suppose the opposite, i.e. some point $x > \frac{r+\lambda}{\lambda} K$ is not optimal near $\tau = 0$. Using that $F(t,x) > N(t,x)$ in the continuation region and the Black-Scholes equation which holds in this region, we obtain

$$\begin{aligned} 0 &< \lim_{t \to T} \frac{F(t,x) - N(t,x)}{T-t} \\ &= -\lim_{t \to T} \frac{F(T,x) - F(t,x)}{T-t} + \lim_{t \to T} \frac{N(T,x) - N(t,x)}{T-t} \\ &= \mathcal{A}F(T,x) - rF(T,x) + N_t(T,x) \\ &= e^{-\lambda t}\left[(r+\lambda)K - \lambda x\right] < 0. \end{aligned} \tag{5}$$

The contradiction proves the desired result. □

3.2. Perpetual values

Suppose now that there are no maturity constraints. Hence, the optimal boundaries are time independent. Let us define the constants p and q as

$$\begin{aligned} p &= 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \\ q &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \frac{r}{\sigma^2} - \frac{1}{2}. \end{aligned} \tag{6}$$

Let ζ be the first hitting moment of the underlying asset to a value c. Note that we can view it as the first hit of the Brownian motion to the linear function $d(t) = d_1 t + d_2$, where

$$\begin{aligned} d_1 &= \frac{\sigma}{2} - \frac{r}{\sigma} \\ d_2 &= \frac{1}{\sigma} \ln\left(\frac{c}{x}\right). \end{aligned} \tag{7}$$

We need the following lemma.

Lemma 1. *Let the underlying asset starts from the value x and $c > x$. If k and θ are constants such that $k < -\frac{\theta^2}{2}$, then*

$$\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta}\right] = 0. \tag{8}$$

On the other hand, if $d_1 > \theta$ and $k = -\frac{\theta^2}{2}$, then

$$\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2d_2(\theta - d_1)}. \tag{9}$$

Note that the notation E^x means the expectation under the assumption that the underlying asset's initial value is x.

Proof. We obtain the desired results using theorem 3.2 from [27] for $z = -\infty$ which leads to

$$E^x \left[e^{\theta B_T} I_{T<\zeta}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(\frac{d(T) - T\theta}{\sqrt{T}}\right) - e^{2d_2(\theta-d_1)} N\left(\frac{d(T) - T\theta - 2d_2}{\sqrt{T}}\right)\right]. \tag{10}$$

□

The following propositions give the closed form formulas for the boundaries as well as for the prices.

Proposition 2. *The optimal boundary of a long futures contract is*

$$c(\infty) = \frac{p-q}{p-q-1} K. \tag{11}$$

The fair price is

$$F(x) = \left(\frac{x}{p-q}\right)^{p-q} \left(\frac{p-q-1}{K}\right)^{p-q-1} \tag{12}$$

when $x < c(\infty)$ and $F(x) = x - K$ otherwise.

Proof. Suppose that the holder exercises when the asset reaches a large enough value c. Using the equality

$$E\left[e^{-y\zeta} I_{\tau<\infty}\right] = e^{-\left(\sqrt{d_1^2+2y}+d_1\right)d_2}, \tag{13}$$

see for example [4] page 223, (2.0.1), we derive

$$\begin{aligned} F(x;c) &= E^x\left[e^{-(r+\lambda)\zeta}(S_\zeta - K) I_{\zeta<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T}(S_T - K) I_{T<\zeta}\right] \\ &= (c-K)\left(\frac{x}{c}\right)^{p-q} + x\lim_{T\to\infty} e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T} E^x\left[e^{\sigma B_T} I_{T<\zeta}\right] = (c-K)\left(\frac{x}{c}\right)^{p-q}. \end{aligned} \tag{14}$$

Note that the limit above is zero due to Lemma 1. We can easily check that function (14) achieves its maximum namely for c given by formula (11) and it leads to price (12). □

Proposition 3. *If we have a short futures, then*

$$c(\infty) = \frac{q}{q+1} K. \tag{15}$$

Its price is

$$F(x) = \left(\frac{K}{q+1}\right)^{q+1} \left(\frac{q}{x}\right)^{q} \tag{16}$$

when $x > c(\infty)$ and $F(x) = K - x$ otherwise.

Proof. Working similarly to the proof of proposition (2), having in mind that the Brownian motion is a symmetric process, and choosing c to be small enough, we derive

$$F\left(x;c\right)=\frac{\left(K-c\right)c^{q}}{x^{q}}-x\lim_{T\to\infty}e^{-\left(\lambda+\frac{\sigma^{2}}{2}\right)T}E^{x}\left[e^{\sigma B_{T}}I_{T<\zeta}\right]. \tag{17}$$

The limit above is zero because of $\lambda>0$ and Lemma 1. We finish the proof observing that function (17) has a maximum for c given by (15). □

Remark 2. *Let us check what happens when $\lambda=0$. In this case, the limit in formula* (17) *is not zero, but $1-\left(\frac{c}{x}\right)^{q+1}$ due to equation* (9)*. Note that we need to use again its symmetrical version (it holds for $d_1<\theta$) since $c<x$ for the short positioned futures. Thus, price* (17) *turns to $F\left(x;c\right)=K\left(\frac{c}{x}\right)^{q}-x$ and its maximum in the interval $\left(0,x\right]$ is for $c=x$.*

3.3. Finite maturity

We shall approximate the optimal boundary by exponents of piece-wise linear functions. Let us divide the time to maturity interval $[0,T]$ into l sub-intervals – $0\equiv t_0<t_1<...<t_l\equiv T$ and suppose that the holder exercises when the underlying asset reaches the level $\exp\left(a_it+b_i\right)$ if this happens in the interval $[t_{i-1},t_i)$. We impose a continuity at the nodes, i.e. $\exp\left(a_it_i+b_i\right)=\exp\left(a_{i+1}t_i+b_{i+1}\right)\equiv c_i$. Note that this strategy can be viewed as the Brownian motion's first hitting to the level $C_it_i+D_i$ for

$$\begin{aligned}C_i&=\frac{1}{\sigma}\left(a_i-r+\frac{\sigma^2}{2}\right)\\ D_i&=\frac{b_i-\log\left(x\right)}{\sigma}.\end{aligned} \tag{18}$$

We can derive the price of a long futures as

$$\begin{aligned}F\left(x;\left\{t_0,...,t_l\right\};\left\{c_0,...,c_l\right\}\right)&=E^{x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta-K\right)I_{\zeta<T}\right]+E^{x}\left[e^{-(r+\lambda)T}(S_T-K)I_{\zeta\geq T}\right]\\ &=x\sum_{m=1}^{l}e^{\sigma D_m}E\left[e^{-\alpha_{2,m}\zeta}I_{t_{m-1}<\zeta\leq t_m}\right]-KE\left[e^{-\alpha_1\zeta}I_{\zeta<T}\right]\\ &\quad+xe^{-\alpha_3T}E\left[e^{\sigma B_T}I_{\zeta\geq T}\right]-Ke^{-\alpha_1T}Q\left(\zeta\geq T\right)\end{aligned} \tag{19}$$

for

$$\begin{aligned}\alpha_1&=r+\lambda\\ \alpha_{2,m}&=\frac{\sigma^2}{2}-C_m\sigma+\lambda\\ \alpha_3&=\lambda+\frac{\sigma^2}{2}.\end{aligned} \tag{20}$$

If we have a short position, then the hitting is to a boundary below and thus formula (19) turns to

$$\begin{aligned}F\left(x;\left\{t_0,...,t_l\right\};\left\{c_0,...,c_l\right\}\right)=KE\left[e^{-\alpha_1\zeta}I_{\zeta<T}\right]-x\sum_{m=1}^{l}e^{\sigma D_m}E\left[e^{-\alpha_{2,m}\zeta}I_{t_{m-1}<\zeta\leq t_m}\right]\\ +Ke^{-\alpha_1T}Q\left(\zeta\geq T\right)-xe^{-\alpha_3T}E\left[e^{\sigma B_T}I_{\zeta\geq T}\right].\end{aligned} \tag{21}$$

The corresponding expectations are derived in [27]. We approximate the optimal boundary backwards. Its value at the maturity is given in Proposition 1. Suppose that we have derived all boundary values after some moment $m \leq l$. The long boundary at the previous node is obtained as the lower value of c for which the payment function

$$h(c) = F(x; \{0, t_m - t_{m-1}, ..., t_l - t_{m-1}\}; \{c, c_m, ..., c_l\}) \tag{22}$$

given in equation (19) has a maximum for $c = x$. In fact, this is the lower value for which the immediate exercise is optimal. For short positions we search for the higher such value using function (21).

4. Pricing

We have already derived closed-form formulas for the perpetual prices in Propositions 2 and 3. Suppose that the maturity is finite and we have approximated the optimal boundary. Hence, we can view the pricing problem as a boundary value problem (BVP). It has the following form for the long positions

$$\begin{aligned}
&F_t(t,x) + rxF_x(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) - rF(t,x) = 0 \\
&F(t,0) = -e^{-r(T-t)}e^{-\lambda T}K \\
&F(t,c(t)) = e^{-\lambda t}(x-K), \quad t \in (0,T) \\
&F(T,x) = e^{-\lambda T}(x-K), \quad x \in (0,c(0)).
\end{aligned} \tag{23}$$

If we have a short position, then the BVP turns to

$$\begin{aligned}
&F_t(t,x) + rxF_x(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) - rF(t,x) = 0 \\
&F(t,c(t)) = e^{-\lambda t}(K-x), \quad t \in (0,T) \\
&F(T,x) = e^{-\lambda T}(K-x), \quad x \in (c(0),\infty).
\end{aligned} \tag{24}$$

In this case, the continuation region is open above. To overcome this problem, we introduce a large enough auxiliary upper boundary $\overline{C}$. The value of the price function on it can be approximated as

$$\begin{aligned}
F\left(t;\overline{C}\right) &= E^{t,\overline{C}}\left[e^{-r(T-t)}e^{-\lambda T}(K-S_T)\right] \\
&= e^{-r(T-t)}e^{-\lambda T}K - \overline{C}e^{\left(-\frac{\sigma^2}{2}\right)(T-t)-\lambda T}H\left(\sigma\sqrt{T-t}\right),
\end{aligned} \tag{25}$$

where $H(\cdot)$ is the moment generating function of the standard normal distribution. We decided to use the Crank-Nicolson finite difference method to solve the BVPs. It is based on the division of the space w.r.t. the time as well as w.r.t. the state (price) variable. Let us notate the values of the price function by $F(m,n)$ – m is for the time and n is for the state variable. We work backwards – note that the prices at the maturity are available. If we know all values for some moment, we approximate the derivatives by the formulas

$$\begin{aligned} F_t &= \frac{F(m-1,n)-F(m,n)}{\Delta t} \\ F &= \frac{F(m-1,n)+F(m,n)}{2} \\ F_x &= \frac{F(m-1,n)-F(m-1,n-1)+F(m,n)-F(m,n-1)}{2\Delta x} \\ F_{xx} &= \frac{F(m-1,n+1)-2F(m-1,n)+F(m-1,n-1)}{2(\Delta x)^2} \\ &+ \frac{F(m,n+1)-2F(m,n)+F(m,n-1)}{2(\Delta x)^2}. \end{aligned} \tag{26}$$

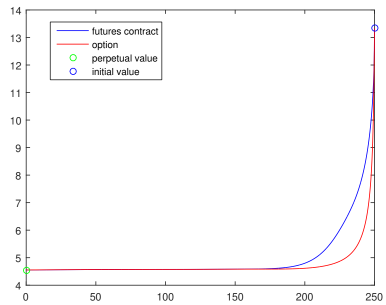
The BVP (23) can be written as

$$\begin{aligned} 0 =& \frac{F(m-1,n)-F(m,n)}{\Delta t} + \\ &+ \frac{1}{2} r x_n \frac{F(m-1,n)-F(m-1,n-1)+F(m,n)-F(m,n-1)}{\Delta x} \\ &+ \frac{1}{4}\sigma^2 x_n^2 \left(\begin{array}{l} \frac{F(m-1,n+1)-2F(m-1,n)+F(m-1,n-1)}{(\Delta x)^2} \\ + \frac{F(m,n+1)-2F(m,n)+F(m,n-1)}{(\Delta x)^2} \end{array} \right) \\ &- \frac{1}{2} r \left(F(m-1,n)+F(m,n)\right). \end{aligned} \tag{27}$$

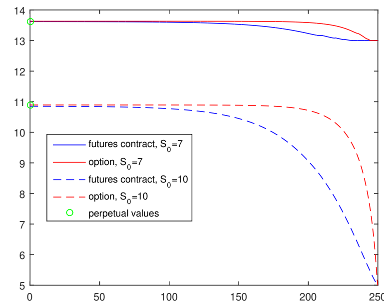
We reach to a linear system for the price function at the previous moment and the state points belonging to the continuation region. We derive the values solving this system.

5. Numerical results. American options vs. American futures contracts

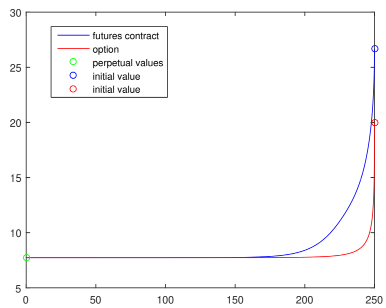
We shall examine the differences appearing between the American options and the corresponding futures contracts. The results of Section 3.2 and section 6 of [28] show that the perpetual boundaries' values coincide. The same is true for the initial values when $r < 0$ for a put option and a short futures and when $r > 0$ for a call option and a long futures. We provide several experiments using the following values of the parameters – the risk-free rate is $r = \pm 0.01$, the additional discount factor is assumed to be $\lambda = 0.03$, the volatility is $\sigma = 0.3$, and the delivery (strike) price is $K = \$20$. We first consider short positions. The boundaries for a negative risk-free rate ($r = -0.01$) are presented in Figure 1a – the blue line is the futures boundary and the red one is the optimal boundary of the related American put. The initial and perpetual values coincide – they are $c(0) = 13.3333$ and $c(\infty) = 4.5353$ and we mark them by green and blue colors, respectively. The price behavior can be viewed in Figure 1b. The initial asset price is assumed to be $S_0 = \$7$ (the solid lines) or $S_0 = \$15$ (the dashed lines). It can be seen that the perpetual prices are equal, namely \$13.6164 when $S_0 = \$7$ and 10.8891 when $S_0 = \$15$. Let us mention that the initial prices are chosen in a manner that makes the comparison most informative regardless of whether they are below or above the delivery (strike) price. Note that both initial prices are below the strike in the current case, whereas one of them is below and the other is above the strike in the next examples.



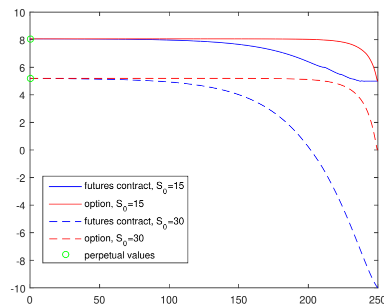
(a) *Short boundary,* $r = -0.01$



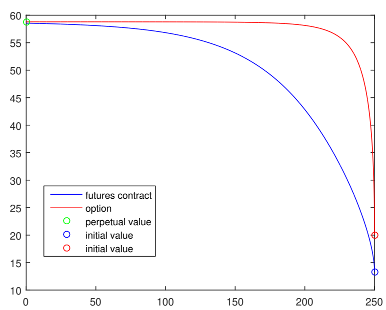
(b) *Short prices,* $r = -0.01$



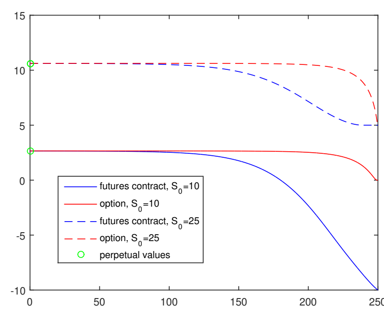
(c) *Short boundary,* $r = 0.01$



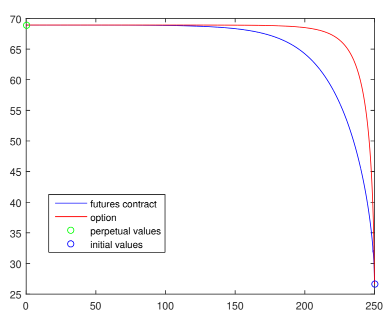
(d) *Short prices,* $r = 0.01$



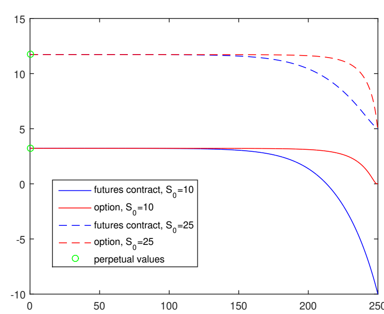
(e) *Long boundary,* $r = -0.01$



(f) Long price, $r = -0.01$



(g) *Long boundary,* $r = 0.01$



(h) *Long price,* $r = 0.01$

Figure 1: *Optimal boundaries and prices.*

The optimal boundaries when the risk-free rate is positive ($r = 0.01$) are presented in Figure 1c. The perpetual values coincide again and they are 7.7374. On the opposite, the initial boundary value for the option is the strike and it is lower than the futures contract's value, which is 26.6667. The prices can be seen in Figure 1d. The initial asset prices used are \$15 (the solid line) and \$30 – the dashed one. The perpetual prices are equal – \$8.0757 and \$5.2148, respectively.

Next, we discuss the long positions. The boundaries for $r = -0.01$ are presented in Figure 1e. As we mentioned above, if the risk-free rate is negative then the perpetual values coincide – their values are $c(\infty) = 58.7980$ – but the initial values differ – the option's one is the strike, whereas the value for the futures contract is $c(0) = 13.3333$. The price behavior is presented in Figure 1f – the solid lines correspond to an initial value $S_0 = \$10$ whereas the dashed ones are for $S_0 = \$25$. The perpetual prices are \$2.6476 and \$10.6151, respectively. We can see that the options' prices are larger due to their higher payments.

Let us consider a positive risk-free value, $r = 0.01$. The optimal boundaries can be viewed in Figure 1g. The option and the futures contract have the same initial and perpetual values – they are $c(0) = 26.6667$ and $c(\infty) = 68.9293$. The prices are presented in Figure 1h – the meaning of the solid and dashed lines is preserved. The perpetual prices are \$3.2245 when $S_0 = \$10$ and \$11.7237 when $S_0 = \$25$.

6. Conclusions

The American style futures contracts are considered in the present study. The long and short positions are examined separately in relation with the corresponding call and put options. The results we derived are in several directions. First, the optimal boundaries are approximated and a pricing method is provided. Closed-form formulas are obtained in the perpetual case. It turns out that the immediate exercise of a long futures is preferable for large enough values of the underlying asset – the same is true for the call options too. Also, the critical values for the futures are lower than the corresponding ones for the call. On the contrary, the lower levels for the underlying asset make the short futures and the put option preferable for immediate exercise and the critical values for the futures are larger. We can conclude that the holder is willing to wait more time to exercise when the derivative is an option. This way the optimal boundary for a call option is above the boundary for the related long futures. The opposite is true when the option is of the put style and the futures is short positioned. Also, the options' prices are larger due to their higher payments.

The results obtained can be useful for the financial practice in several directions. First, an investor can assure his position paying a lower price using a futures contract instead of an option, because, as a rule, the futures are cheaper. The price of this lower cost is the risk for the holder to sell/buy the underlying asset at a potential loss at the maturity date. Note that this loss is impossible for the options since their payoffs are constrained to be non-negative. Second, the American features gives the holder the right to optimize the moment at which to execute the trade, thus maximizing his/her financial utility. Last but not least, the proposed approach based on finding the optimal exercise boundary can be useful in the decision-making task, because at every moment the investor knows whether it is favorable to exercise or not.

Acknowledgments

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0008.

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