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# Characterizing fixed points*

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**Abstract.** A set of sufficient conditions which guarantee the existence of a point $x^\star$ such that $f(x^\star) = x^\star$ is called a "fixed point theorem". Many such theorems are named after well-known mathematicians and economists.
Fixed point theorems are among most useful ones in applied mathematics, especially in economics and game theory. Particularly important theorem in these areas is Kakutani's fixed point theorem which ensures existence of fixed point for point-to-set mappings, e.g., [2, 3, 4]. John Nash developed and applied Kakutani's ideas to prove the existence of (what became known as) "Nash equilibrium" for finite games with mixed strategies for any number of players. This work earned him a Nobel Prize in Economics that he shared with two mathematicians. Nash's life was dramatized in the movie "Beautiful Mind" in 2001.
In this paper, we approach the system $f(x) = x$ differently. Instead of studying existence of its solutions our objective is to determine conditions which are both necessary and sufficient that an arbitrary point $x^\star$ is a fixed point, i.e., that it satisfies $f(x^\star) = x^\star$. The existence of solutions for continuous function $f$ of the single variable is easy to establish using the Intermediate Value Theorem of Calculus. However, characterizing fixed points $x^\star$, i.e., providing answers to the question of finding both necessary and sufficient conditions for an arbitrary given $x^\star$ to satisfy $f(x^\star) = x^\star$, is not simple even for functions of the single variable. It is possible that constructive answers do not exist. Our objective is to find them.
Our work may require some less familiar tools. One of these might be the "quadratic envelope characterization of zero-derivative point" recalled in the next section. The results are taken from the author's current Research project "Studying the Essence of Fixed Points". They are believed to be original. The author has received several feedbacks on the preliminary report and on parts of the project which can be seen on Internet [9].

**Keywords**: single variable calculus, fixed point, fundamental theorem of calculus, quadratic envelope characterization of zero-derivative point, primal and dual characterizations of fixed points

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## 1. Introduction

A set of sufficient conditions which guarantee the existence of a point $x^\star$ such that $f(x^\star) = x^\star$ is called a "fixed point theorem". Our objective is not to come up with another such theorem. Instead we wish to characterize points $x$ for which $f(x) = x$. This will be done at primal and dual levels, producing different geometric interpretations. At the primal level these points will turn out to be apexes of certain classes of parabolas. At the dual level they are characterized by uniformly bounded ratios of functions around $x^\star$ but not at $x^\star$ itself.
The results are proved using two major theorems: quadratic envelope characterization of zero-derivative points and the fundamental theorem of calculus. Here is the former.

**Theorem 1** (Quadratic envelope characterization of zero-derivative points [7]). *Consider a continuously differentiable function of the single variable with Lipschitz derivative on an interval $I = [a, b]$. If $a < x^\star < b$, then $f'(x^\star) = 0$ if, and only if there is a constant $\Lambda \geq 0$ such that*

$$|f(x) - f(x^\star)| \leq \Lambda(x - x^\star)^2,$$

*for every $x \in I$.*

Note that this theorem talks about zero-derivative points without using differentiation. It was proved for functions in n variables in [7]. One can find it depicted in this author's data "Formula" on Researchgate. Its simplified proof for $n = 2$ and for $C^2$ functions is given in the textbook [6]. Various discussions regarding this result can be found in the Q&A section on Researchgate under the question "Is there a book in English ..." We have not yet seen an affirmative answer to this question.
The interested reader can find more on fixed points in, e.g., [1, 2, 5] and Journal on Fixed Point Theory and Applications. Depictions of fixed points in one, two and three dimensions using string, disc and a cup of coffee, respectively, can be found in the literature typically related to the Brouwer fixed point theorem.

## 2. Primal characterization of fixed points

In this section we characterize fixed points using a particular integral. Since the results appear to be new, and possibly non-intuitive, we illustrate them by elementary examples.

**Theorem 2** (Primal characterization of fixed points). *Consider a continuous Lipschitz function $f$ of the single variable $x$ on $I = [a, b]$ and a point $x^\star$ such that $a < x^\star < b$. Denote*

$$W(x) = \int_{x^\star}^{x} (f(t) - t)\, dt, \quad \text{on } [x^\star, x].$$

*Then $f(x^\star) = x^\star$ if, and only if*

$$|W(x)| \leq \Lambda(x - x^\star)^2, \quad x \in I, \tag{1}$$

*for some $\Lambda \geq 0$.*

**Proof**. The derivative of $W$ is $W'(x) = f(x) - x$ by the Fundamental theorem of calculus. Hence $x^\star$ is a zero-derivative point of $W(x)$. Using the quadratic envelope property of $C^1$ functions with Lipschitz derivative (Theorem 2), we have

$$|W(x) - W(x^\star)| \leq \Lambda(x - x^\star)^2, \tag{2}$$

for every $x$ in $I$ and some $\Lambda \geq 0$. But $W(x^\star) = 0$, yielding (1).
Conversely, we assume that (1) holds. We know that $W(x^\star) = 0$, hence (2) holds. Hence, $W(x^\star) = 0$ by Theorem 1. On the other hand, the fundamental theorem of calculus gives $W(x) = f(x) - x$, for $a < x < b$. Hence $f(x^\star) = x^\star$. □

We note that if some $\Lambda$ over-estimates $W(x)$ on $I$ as in (1), then so does every bigger $\Lambda$. Therefore we can talk about a class of ("sufficiently large") parabolas. Theorem 2 has an interesting geometric interpretation:

**Corollary 1.** *Let $f$ and $I$ be as above. If $a < x^\star < b$ is a fixed point $f(x^\star) = x^\star$, then $x^\star$ is the apex of a class of parabolas over-estimating the absolute value of the function $W(x)$ on $I$.*

Let us illustrate this result by elementary examples.

**Example 1.** *Consider interval $I = [-1, 1]$ and the fixed point $x^\star = 0$ of the function $f(x) = \sin x$. Then*

$$|W(x)| = |(\cos x + \frac{1}{2}x^2 - 1)| \leq \Lambda x^2,$$

*for every $x$ in $I$ and some $\Lambda \geq 1$.*
*Indeed, $x^\star$ is an apex of a class of "suitable" parabolas $rx^2$ on $I$ for all sufficiently large $r$, e.g., $r \geq 1$. Note that, in this example, the graph of the absolute value of $W(x)$ lies just above the $x$ axis. (If $x^\star$ were not a fixed point, such class would not have existed; compare Figures 1 and 2. The two parabolas in Figure 1 correspond to "large enough" $\Lambda$'s.)*

**Example 2** (Trivial example)**.** *Consider the zero function $f(x) = 0$ on $I = [-1, 1]$. Could $x^\star = 0$ be a fixed point of $f(x)$? Affirmative, if by the theorem $|x + x^\star| \leq 2\Lambda|x - x^\star|$ on $I$ for some $\Lambda \geq 0$. True.*

**Example 3.** *Is $x^\star = 0$ a fixed point of $f(x) = 1$ on $I = [-1, 1]$? We use Theorem 2. Point $x^\star$ is not an apex of any class of parabolas over-estimating $|W(x)| = |x - \frac{1}{2}x^2|$ on $I$, so the answer is negative.*

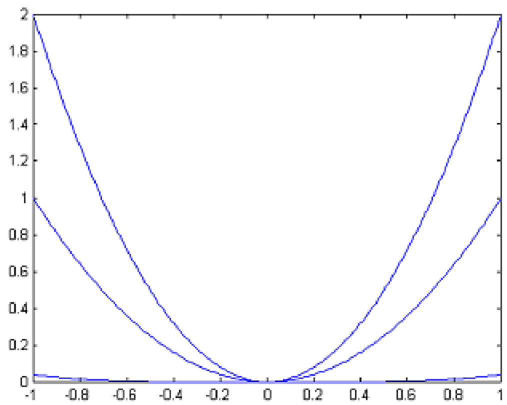


Figure 1: *Primal condition for fixed point $x^\star = 0$ holding*

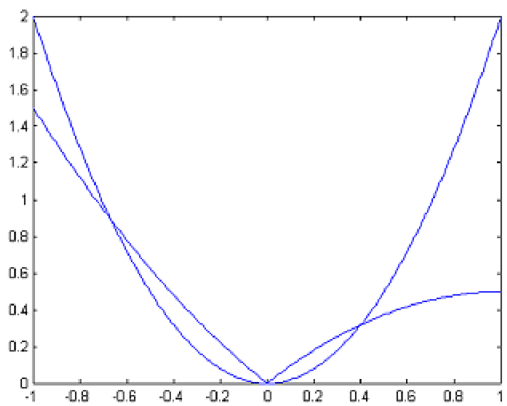


Figure 2: *Violation of the primal necessary condition for fixed point*

## 3. Dual characterization of fixed point

We know that for fixed points there exist positive $\Lambda$'s in (1). After division by any of such numbers we obtain "dual" characterization of fixed points. In order to formulate these, we use functions which are possibly not defined at $x^\star$ in $I$ but which are uniformly bounded on the complementary set $I\backslash\{x^\star\}$. Such functions were used in a different context in, e.g., [8].

**Theorem 3** (Dual characterization of fixed point)**.** *Consider a continuous Lipschitz function $f$ of the single variable $x$ on $I = [a, b]$ and a point $x^\star$ such that $a < x^\star < b$. Recall $W(x)$ and denote*

$$R(x) = \frac{|W(x)|}{(x - x^\star)^2} \quad \textit{on } I\backslash\{x^\star\}.$$

*Now $f(x^\star) = x^\star$ is a fixed point if, and only if $R(x)$ is bounded on $I\backslash\{x^\star\}$ by some* *constant* $\Lambda \geq 0$.

**Example 4.** *We know that* $x^\star = 0$ *is a fixed point of* $f(x) = \sin x$ *on* $I = [-1, 1]$. *Therefore,*

$$R(x) = \frac{|\cos x + \frac{1}{2}(x^2 - 1)|}{x^2}$$

*is bounded by some constant on* $I \backslash \{0\}$. *The graph of such R(x) is depicted in Figure 3.*

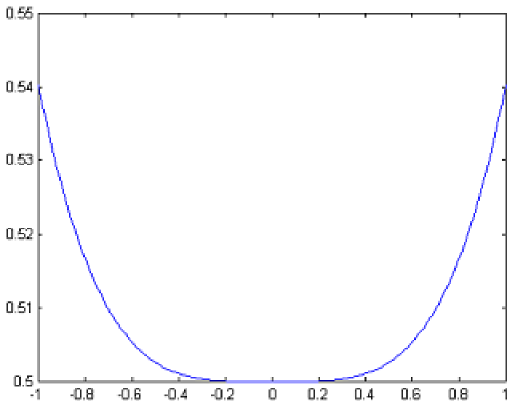


Figure 3: *Dual necessary condition for fixed point* $x^\star = 0$

**Example 5.** *Consider* $I = [-1, 1], f(x) = 0$ *and* $x^\star = 0$. *Around the point* $x^\star$ *we find, after division, that* $R(x) = \frac{|x|}{|x|} = 1, x \neq 0$, *is bounded so* $x^\star$ *is a fixed point.*

**Example 6.** *However,* $x^\star = 0$ *is not a fixed point of the function* $f(x) = 1$ *on* $I = [-1, 1]$ *because* $R(x) = \frac{|x - x^2|}{x^2}$ *is unbounded on* $I \backslash \{x^\star\}$. *Depicted in Figure 4.*

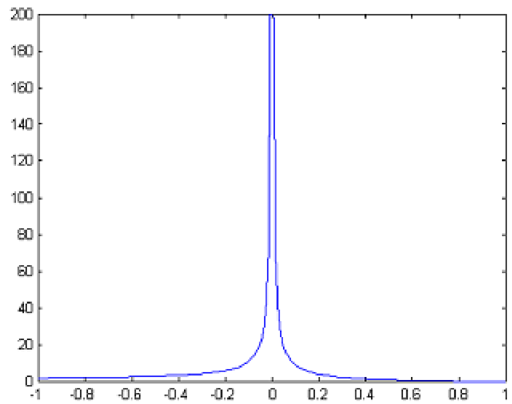


Figure 4: *Violation of the dual necessary condition for fixed point*

## 4. Conclusion

We study functions of the single variable and find conditions, for a given arbitrarily chosen point $x^\star$, which are both necessary and sufficient for a fixed point. This is done at two levels. At the primal level we show that a fixed point is an apex of a particular class of parabolas. At the dual level, fixed point is characterized by boundedness of particular ratio functions in a neighbourhood of $x^\star$ but not at $x^\star$ itself. It is expected that the results given hereby will possibly lead to new directions in the study of fixed points for functions of several variables and advance the study of equilibria in the theory of games, economics and other areas.

## Acknowledgement

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