A DEA-based Approach for Finding a Favorable Multi-Objective Shortest Path

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Abstract. Multi-objective shortest path problems deal with networks in which each are is associated with multiple types of costs and profits. The conditions can increase number of non-dominated paths and compounds difficulty of selecting a favorable one for Decision Maker (DM). In this paper we suggest a hybrid method to generate a non-dominated path which fulfills decision maker's demands. It uses DEA in a more logical way than some previous DEA-based approaches. We take advantage of common weights methodology. DM can give relative importance of some cost and profit indicators. Eventually, we have a shortest path problem that can be solved with Dijkstra's algorithm. Two numerical examples are presented and the results are analyzed.

1. Introduction

The shortest path problem is one of the most important problems in network flows with wide range of applications in different areas (telecommunications, transportations, manufacturing, etc.). In these problems, each arc has just one cost. For example, if an arc represents a road, this cost can be the spent money or the amount of time needed for passing the road. The objective is to find a path with minimum cost or a path with fewest arcs; in traditional form of the shortest path problem, there is only one objective function. There are many algorithms to solve this problem including works of Dijkstra et al. [12] and Ahuja et al. [2]. But in real world, it is very likely that different types of costs be allocated to one arc. For example, in arc-road correspondence, one can divide the cost to spent money and time and add some profits like security, desirability, etc. Therefore, for moving from a node to another, we want to find a path which is the best solution respect to multiple objectives. That is why these problems are called

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multi-objective shortest path problems. In most cases, there is no optimal solution; therefore one seeks non-dominated (Pareto-optimal) solutions: solutions that cannot be replaced with the better ones. Martins et al. [20] presented two algorithms for MCSP problem. One of them was a generalization of multiple labeling schemes algorithm of Hansen [17] for the bi-criteria case, and the second one was the simplex method used in continuous linear multi-objective programming. There are some works as those have done by Guerriero et al. [15], Brumbaugh-Smith et al. [5], Mote et al. [23], Bertsekas et al. [4], Climaco et al. [8], Hamacher et al. [16], Current et al. [11], Modesti et al. [22], Coutinho-Rodrigues et al. [10], Murthy et al. [24], Duque et al. [13], Sedeno-Noda et al. [26], Sinha et al. [27]. DEA, firstly introduced by Charnes et al. [6], is a non-parametric method to assess relative performance of a group of homogeneous Decision Making Units (DMUs), it's explained more latter. Recently, some DEA-based methods have been suggested to generate non-dominated paths. Masoumi et al. [21] let each path from the source node to the sink one to be a DMU that had no output and with costs as input. They showed each non-dominated path is equivalent to a strongly efficient DMU and used FDH model. Amirteimoori [3] extended the classical shortest path problem by considering multiple attributes for each arc (as inputs and outputs). He obtained two measures for each arc by using DEA models and associated the average of two measures to the arc. Then, he defined the overall efficiency of each path, from the source node to the sink one. Finally, the efficient path, a path with maximum efficiency, was determined.

Moreover, some methods with the same approach have been proposed in assignment problem, we can mention some works as those have done by Zarafat Angiz et al. [29], Chen et al. [7], Emrouznejad et al. [14]. In this paper, we suggest a method where DM's points of view and DEA methodology will help us to find a non-dominated multiple objective shortest path. To accomplish that, first we convert DM's propensities to restrictions about weight of indicators and then we add them to a DEA model for generating a Common Set of Weights (CSW). Our purpose is to generate a CSW coming from involvement of DM's preferences and fairness in evaluation of all arcs. The latter is what puts distinction between our method and some other DEA-based methods where each arc or path is evaluated by its own weight. As matter of fact, it seems more reasonable that arcs and paths to be assessed under the same conditions. In our DEA-based method, we define a DMU for each arc. Since our overall intention is to decrease cost and increase profit, therefore cost and profit of an arc are assumed to be respectively input and output. Then, based on generated CSW, a virtual scalar cost is allocated to each arc. It transforms the problem to a simple shortest path problem. It turns out that the scalar criterion is non-negative and Dijkstra's algorithm or any shortest path algorithm can be used to solve it. This is essential from computational complexity angle. At the end, we prove that the path is non-dominated. In this paper, we briefly review the literature and required notions in section 2 and introduce the main problem in section 3. The proposed model, including its details, is discussed in section 4. We, also, bring two numerical examples, discuss and compare their results in section 5.

2. Background

In this section, we mention some necessary definitions and theorems.

2.1. Multi objective linear programming problem

The general form of a Multi-objective Linear Programming problem can be represented as following:

$$Max \{f_1(\boldsymbol{x}), \dots, f_l(\boldsymbol{x})\} \tag{1}$$
$$Min \{ h_1(\boldsymbol{x}), \dots, h_f(\boldsymbol{x})\}$$
$$s.t.$$
$$\boldsymbol{x} \epsilon S$$

where x represents the vector of decision variables, $f_r(\boldsymbol{x})$ $(r = 1, \dots, l)$ and $h_i(\boldsymbol{x})$ $(i = 1, \dots, f)$ are linear objective functions, $S = \{\boldsymbol{x} | g_j(\boldsymbol{x}) \leq b_j, j = 1, \dots, k\}$ and $g_j(\boldsymbol{x})(j = 1, \dots, k)$ are linear functions and $b_j \epsilon \mathbb{R}$ $(j = 1, \dots, k)$.

A non-dominated solution (Pareto-optimal solution) for the above problem defines as following:

Definition1: $\bar{x} \epsilon S$ *is a non-dominated (Pareto-optimal) solution for (1) if it satisfies following conditions:*

$$\exists! x \epsilon S : (f_r(x) \geq f_r(\overline{x}) (r = 1, \dots, l), h_i(x) \leq h_i(\overline{x}) (i = 1, \dots, f), \tag{2}$$

where there is at least one strict inequality.

The following Theorem states one of methods for finding a non-dominated solution to (1):

Theorem 1. *Let* $\alpha_r \epsilon \mathbb{R}^+ (r = 1, \dots, l)$ *and* $\beta_i \epsilon \mathbb{R}^+$ $(i = 1, \dots, m)$. *Then, the optimal solution of (3) is a non-dominated solution for (1).*

$$Min \ \ y(\boldsymbol{x}) = \sum_{i=1}^{f} \beta_i h_i \ (\boldsymbol{x}) - \sum_{r=1}^{l} \alpha_r f_r \ (\boldsymbol{x}) \tag{3}$$
$$s.t.$$
$$\boldsymbol{x} \epsilon S$$

Proof. [18]

2.2. Data Envelopment Analysis

DEA is a quantitative non-parametric method that uses mathematical programming for measuring and evaluating relative performance of a group of homogeneous Decision Making

Units with the same types of inputs and outputs. Therefore, it has grown into a powerful management tool in operations research and economics.

Let x_{ij} denotes the data of ith $(i = 1, \dots, f)$ input for DMU_j and y_{ij} denotes the data of rth $(r = 1, \dots, l)$ output for it.

Let $(\boldsymbol{u}, \boldsymbol{v})$ be an arbitrary set of weights where $\mathbf{u}^t \epsilon \mathbb{R}^l, \mathbf{v}^t \epsilon \mathbb{R}^f$.

The absolute efficiency of DMU_o $(o \epsilon \{1, \dots, n\})$, based on $(\boldsymbol{u}, \boldsymbol{v})$, is:

$$\frac{\sum_{r=1}^{l} u_r y_{ro}}{\sum_{i=1}^{f} v_i x_{io}} \tag{4}$$

The (5) shows multiplier form of input oriented CCR model for evaluating DMU_o is as follows:

$$\theta_0^* = Max \ \boldsymbol{uy_o} \tag{5}$$
$$\text{s.t.}$$
$$\boldsymbol{vx_o} = 1$$
$$\boldsymbol{uy_j} - \boldsymbol{vx_j} \leq 0 \quad j = 1, \dots, n$$
$$\boldsymbol{u} \geq 0, \boldsymbol{v} \geq \boldsymbol{o}$$

where, the decision variables $\boldsymbol{v} \epsilon \mathbb{R}^f$ and $\boldsymbol{u} \epsilon \mathbb{R}^l$ represent respectively the input weights and output weights. $\boldsymbol{x_j} \epsilon \mathbb{R}^f$ and $\boldsymbol{y_j} \epsilon \mathbb{R}^l$ $(j = 1, \dots, n)$ indicate respectively the vector of inputs and the vector of outputs for DMU_j.

Dual of **(5)** is **(6)** that is called envelopment form of input oriented CCR model for assessment of DMU_0.

$$\theta_0^* = min \ \theta \tag{6}$$
$$\text{s.t.}$$
$$\sum_{j=1}^{n} \lambda_j \boldsymbol{x_j} \leq \theta \, \boldsymbol{x_o}$$
$$\sum_{j=1}^{n} \lambda_j \boldsymbol{y_j} \geq \boldsymbol{y_o}$$
$$\lambda_j \geq 0 \quad j = 1, \dots, n$$
$$\theta \ is \ free$$

where, the λ_j $(j = 1, \dots, n)$ and θ are decision variables. $\boldsymbol{x_j} \epsilon \mathbb{R}^f$ and $\boldsymbol{y_j} \epsilon \mathbb{R}^l$ $(j = 1, \dots, n)$ indicate respectively the vector of inputs and the vector of outputs for DMU_j. It turns out that $\theta_0^* \leq 1$. If $\theta_0^* = 1$, then DMU_0 is efficient and otherwise inefficient.

In real problems, extra information may be lead to imposing some weight restrictions on DEA models. Assurance region (AR) method, developed by Thompson et al. [28], impose the weight restrictions stating the lower and upper bounds for ratio of some input and output weights.

Numerous types of ranking methods have been suggested by researchers for distinctions among performance of efficient DMUs. Common weights methodology was introduced by Cook et al. [9] and Roll et al. [25]. In general, ranking methods with common set of weights can provide evaluation of units under equal conditions. Therefore, it is fairer. It is also worth to mention that most of these methods are able to rank all the DMUs (e.g. [12]).

3. Problem definition

Suppose $G = (V, A)$ be a directed graph where V and A ,respectively, indicate the set of nodes and the set of arcs in G, $|A|=m$. $(i,j) \epsilon A$ if and only if there is an arc from ith node to jth node in the network. Let $N = (G, c, p, s, t)$ be a network with distinguished nodes s and t as respectively the source node and destination (sink) node. c and p represent the arc weights. In fact, there are a finite number of indicators to evaluate each arc. for conveniences, we call those DM wishes to be increased, profit indicators and those he/she wishes to be decreased, cost indicators.

It's assumed that the indicators satisfy the following assumptions:
- Number and types of allocated indicators are equal for all the arcs.
- There are f cost indicators and l profit indicators.
- When a unit of flow passes through arc $(i,j) \epsilon A$, amount of the k^{th}($k = 1, \dots, f$) type of cost is c_{ij}^k and it is a non-negative integer number.
- When a unit of flow passes through arc $(i,j) \epsilon A$, amount of the r^{th}($r = 1, \dots, l$) type of profit is p_{ij}^r and it is a non-negative integer number.
- $\forall (i,j) \epsilon A \ \left(\exists k \epsilon \{1, \dots f\} : \ c_{ij}^k > 0 \right) and \ (\exists r \epsilon \{1, \dots l\} : \ p_{ij}^r > 0)$.

Let $\mathcal{P}_{st}$ denotes the set of all path from s to t (s-t paths) in graph G and $P \epsilon \mathcal{P}_{st}$ .When a unit of flow passes through $P,$ amounts of costs and profits are respectively, $c_k(P) = \sum_{(i,j) \epsilon A(P)} c_{ij}^k \ (k = 1, \dots, f)$ and $p_r(P) = \sum_{(i,j) \epsilon A(P)} p_{ij}^r \ (r = 1, \dots l)$ where A(P) is the set of arcs in path P.

Definition 2. $\bar{P} \epsilon \mathcal{P}_{st}$ *is a non-dominated (Pareto-optimal) s-t path in graph G if satisfies following conditions:*

$$\exists! P \epsilon \mathcal{P}_{st} : \ (p_r(P) \geq \ p_r(\bar{P}) \ (r = 1, \dots, l), c_i(P) \leq \ c_i(\bar{P}) \ (i = 1, \dots, f), \qquad (7)$$

where there is at least one strict inequality.

Our objective is to find a s-t path so that when a unit of flow passes through it, amount of each indicator be optimum comparing to other s-t paths. It means amount of cost indicators are minimum and the amount of profit indicators are maximum. We call this s-t path an optimal s-t path.

Now we are ready to formulate this Multi Objective Shortest Path Problem as follows:

$$\max\{\textstyle\sum_{(i,j) \in A} p_{ij}^1 x_{ij}, \dots, \ \sum_{(i,j) \in A} p_{ij}^l x_{ij}\} \qquad (8)$$
$$\min\{\textstyle\sum_{(i,j) \in A} c_{ij}^1 x_{ij}, \dots, \ \sum_{(i,j) \in A} c_{ij}^f x_{ij}\}$$
$$\text{s.t.}$$
$$\textstyle\sum_{j \in N^+(i)} x_{ij} - \sum_{j \in N^-(i)} x_{ji} = \begin{cases} 1 & i = s \\ 0 & i \neq s, t \\ -1 & i = t \end{cases}$$
$$x_{ij} \epsilon \{0,1\} \qquad (i,j) \epsilon A$$

where, for each $i \epsilon V$, $N^+(i) = \{j | (i,j) \epsilon A\}$ and $N^-(i) = \{j | (j,i) \epsilon A\}$. All $x_{ij} \epsilon A$ are decision variables. Foe each $(i,j) \epsilon A$, p_{ij}^r $(r = 1, ..., l)$ and c_{ij}^k $(k = 1, ..., f)$ are respectively the profit and cost parameters.

4. A Method for Determining a Favorable Non-dominated s-t Path

Here, we propose a DEA-based approach to introduce a favorable non-dominated s-t path to DM. In comparison with some former works in this area, we attain to utilizing DEA approach in a more logical way. To observe equal and fair conditions, we use common weights methodology to evaluate the arcs of network and consequently the s-t paths. It is also worth in terms of computational aspect. In the following, we elaborate the process.

4.1. Step one: Network as a DEA structure

Consider $N = (G, c, p, s, t)$ with the prior mentioned characteristics (Section 3). Firstly, we correspond each arc $(i,j) \epsilon A$ to a decision making unit DMU_{ij} for which the consumed inputs and produced outputs vectors are equal to the cost and profit vectors of the arc, respectively. That means $[x_{ij}^1, ..., x_{ij}^f] = [c_{ij}^1, ..., c_{ij}^f]$ and $[y_{ij}^1, ..., y_{ij}^l] = [p_{ij}^1, ..., p_{ij}^l]$, where x_{ij}^k $(k = 1, ..., f)$ and y_{ij}^r $(r = 1, ..., s)$ denotes respectively amounts of the ith input and the rth output for DMU_{ij}. Given a set of weighs (u, v), where $u^t \epsilon \mathbb{R}_{\geq 0}^l$ and $v^t \epsilon \mathbb{R}_{\geq 0}^f$, the weighted sum of the inputs (costs), $\sum_{k=1}^f v_k c_{ij}^k$, is considered as a virtual input (cost) and the weighted sum of outputs (profits), $\sum_{r=1}^l u_r p_{ij}^r$, is considered as a virtual output (profit).

4.2. Step Two: Generating a Common Set of Weights

Here, we discuss how a DEA model will be part of finding a favorable s-t path. Definition of the scalar criterion to evaluate arcs (to be discussed later) indicates that the weights used play an essential role in its computation. Hence, we take advantage of common weights methodology in DEA. Also, we want to involve the DM's points of view in process of choosing the CSW and eventually the obtained non-dominated s-t path. Having said that, we explain how to obtain a common set of weights in two steps:

  i. Dealing with DM's Preferences
  ii. Using a DEA model to generate a CSW.

4.2.1. Dealing with DM's Preferences

In our method, DM is asked to determine relative importance of the cost and profit factors, if there are any. He/she is not forced to give an exact value; giving bounds of it (at least one) will work in this case. DM's points of view will appear in the model as weight restrictions. Let i≠j be two arbitrary indicators that DM intends to express his preferences through them. The following formula is generally how we consider the DM's intuition:

$$\xi_{ij} \leq \frac{\textit{weight of ith indicator}}{\textit{weight of jth indicator}} \leq \bar{\xi}_{ij} \tag{9}$$

where satisfies the following:

- It is possible that the indicators be of any type (cost or profit).
- $\xi_{ij} \in \mathbb{R}^+ \cup \{0\}$, $\bar{\xi}_{ij} \in \mathbb{R}^+ \cup \{+\infty\}$
- If DM only determines the lower bound ξ_{ij}, then $\bar{\xi}_{ij}$ is set to $+\infty$.
- If DM only determines the upper bound $\bar{\xi}_{ij}$, then ξ_{ij} is set to be $\mathbf{0}$.
- $\xi_{ij} \leq \bar{\xi}_{ij}$, the equation occurs when DM state the relative importance exactly.

It is worth to mention that DM is supposed to logically assign bounds. It means that, for example, if indicator i is more important than indicator j (for any type of cost or profit) then ξ_{ij} must be greater than 1. Also, at most one formula for each pair of factors is considered. In Section 5, we discuss several different cases and investigate alteration of the parameters and its effects on the final results.

4.2.2. Using a DEA model to generate a CSW

Here, we present a DEA model using assurance region method to generate a CSW. This model has been influenced by Liu and Peng's work [19] which is computationally simple to use. The aim behind the model is to maximize the efficiencies of DMUs by maximizing their summation. The assurance region restrictions carry DM's points of view. The main model to generate a CSW, in a general form, comes as follows:

$$Max \sum_{(i,j)\epsilon A} \frac{\sum_{r=1}^{l} u_r p_{ij}^r}{\sum_{k=1}^{f} v_k c_{ij}^k} \tag{10}$$

$$\text{s.t.}$$
$$\frac{\sum_{r=1}^{l} u_r p_{ij}^r}{\sum_{k=1}^{f} v_k c_{ij}^k} \leq 1 \qquad \forall(i,j)\epsilon A$$
$$\alpha_{kk'} \leq \frac{v_k}{v_{k'}} \leq \bar{\alpha}_{kk'} \qquad \forall(k,k')\epsilon \Gamma_I$$
$$\beta_{rr'} \leq \frac{u_r}{u_{r'}} \leq \bar{\beta}_{rr'} \qquad \forall(r,r')\epsilon \ \Gamma_O$$
$$Y_{rk} \leq \frac{v_i}{u_{r'}} \leq \bar{Y}_{rk} \qquad \forall(r,k)\epsilon \ \Gamma_{IO}$$
$$\Delta_{ij} \geq 0 \qquad \forall(i,j)\epsilon A$$
$$u_r \geq \varepsilon \qquad r=1,...,l$$
$$v_k \geq \varepsilon \qquad k=1,...,f$$

(11) is solved instead of (10) (Refer to [13] for more explanations).

$$Min \sum_{(i,j)\epsilon A} \Delta_{ij} \tag{11}$$

$$\text{s.t.}$$
$$\sum_{r=1}^{s} u_r p_{ij}^r - \sum_{k=1}^{f} v_k c_{ij}^k + \Delta_{ij} = 0 \qquad \forall(i,j)\epsilon A$$
$$\left. \begin{array}{l} -v_k + \alpha_{kk'} v_{k'} \leq 0 \\ v_k - \bar{\alpha}_{kk'} v_{k'} \leq 0 \end{array} \right\} \quad \forall(k,k')\epsilon \ \Gamma_I$$
$$\left. \begin{array}{l} -u_r + \beta_{rr'} u_{r'} \leq 0 \\ u_r - \bar{\beta}_{rr'} u_{r'} \leq 0 \end{array} \right\} \quad \forall(\ r,r')\epsilon \ \Gamma_O$$

$$\left.\begin{array}{l} -u_r + \gamma_{rk} v_k \leq 0 \\ u_r - \bar{\gamma}_{rk} v_k \leq 0 \end{array}\right\} \quad \forall (r,k) \epsilon \Gamma_{OI}$$

$$\Delta_{ij} \geq 0 \qquad\qquad \forall (i,j) \epsilon A$$

$$\left.\begin{array}{ll} u_r \geq \varepsilon & r = 1, \ldots, l \\ v_k \geq \varepsilon & k = 1, \ldots, f \end{array}\right\}$$

where, $v_k \ (k = 1, \ldots, f)$, $u_r \ (r = 1, \ldots, l)$ and $\Delta_{ij} \ (i,j) \epsilon A$ are decision variables. For each $(i,j) \epsilon A, c_{ij}^k \ (k = 1, \ldots, f)$ and $p_{ij}^r \ (r = 1, \ldots, l)$ are respectively cost and profit parameters. The constraints of (1) guarantee that the efficiency of each DMU don't exceed one. The constraints of (2), (3) and (4) state weight restrictions about the relative importance and priority of cost and profit indicators. Γ_I denotes the set of the ordered pairs of input indicators' indexes which DM has given some information about their relative importance. This set satisfy the following properties: if $(i,k) \epsilon \Gamma_I$ then $(k,i) \notin \Gamma_I$; for all $i \epsilon V$, $(i,i) \notin \Gamma_I$. Plus, there are similar statements about Γ_O (related to output indicators' indexes) and Γ_{IO} (related to input and output indicators' indexes). The weight restrictions are given in a general form regarding to (9). The parameters $\alpha_{kk'}$ and $\bar{\alpha}_{kk'}$ ($\forall(k,k') \epsilon \Gamma_I$) are respectively the lower and upper bounds of the relative importance of the input indicator k to k' and the parameters $\beta_{rr'}$ and $\bar{\beta}_{rr'}$ ($\forall(r,r') \epsilon \Gamma_O$) are respectively the lower and upper bounds of the relative importance of the output indicator r to r' . Also, γ_{rk} and $\bar{\gamma}_{rk}$ ($\forall(r,k) \epsilon \Gamma_{IO}$) are respectively the lower and upper bounds of the relative importance of the output indicator r to input indicator k. However, practically, the weight restrictions of (2), (3) and (4) stands only for positive finite bounds (the rest are redundant, and so they are removed). All of these parameters are determined by DM. If he/she has no preference then we do not use the weight restrictions. ε is a positive Archimedean infinitesimal constant. The batch constraints (6) guaranties that all weights in feasible solutions are non-zero.

Assume that (u^*, v^*, Δ^*) be the optimal solution of (11). We accept (u^*, v^*) as the common set of weights. Also, (11) can be rewritten as (12):

$$Min \ \mathbf{1}_m \cdot \Delta \qquad\qquad (12)$$
$$s.t.$$
$$uP - vC + \Delta = \mathbf{0}$$
$$vA \leq \mathbf{0}$$
$$uB \leq \mathbf{0}$$
$$[u,v]C \leq \mathbf{0}$$
$$\Delta \geq 0$$
$$u \geq \varepsilon \mathbf{1}_l$$
$$v \geq \varepsilon \mathbf{1}_f$$

where, $\Delta \epsilon \mathbb{R}^m$, $\mathbf{u}^t \epsilon \mathbb{R}^l (output \ variables), \mathbf{v}^t \epsilon \mathbb{R}^f (input \ variables)$ are vectors of decision variables. C=$[c_{ij}]_{f \times m}$ and P=$[p_{ij}]_{l \times m}$ denote respectively the matrixes of arc costs and arc profits. $A \epsilon \mathbb{R}_{f \times a}$, $B \epsilon \mathbb{R}_{l \times b}$ and $C \epsilon \mathbb{R}_{(l+f) \times h}$ contains the multipliers of assurance region constrains of (11). Regarding to the previous explanations about the bounds of weight ratios derived from DM's preferences, it is evident that: $|\Gamma_I| \leq a \leq 2|\Gamma_I|, |\Gamma_O| \leq b \leq 2|\Gamma_O|$ and $|\Gamma_{OI}| \leq h \leq 2|\Gamma_{OI}|$. For above inequalities, the minimum occur when there is no weight ratio with two positive finite bounds,

and the maximum occur when both of lower and upper bounds of for the weight ratios are positive and finite.

In short, the interaction with DM is placed in this part and it finally leads to choosing a favorable s-t path based on DM preferences.

4.3. Determining the Scalar Criterion

Now, we use a scalar criterion to assess each arc. Let (u^*, v^*) be the generated CSW. We consider the following scalar criterion, obtained based on the CSW, for each arc $(i,j) \in A$:

$$\lambda_{ij} = \sum_{k=1}^{f} v_k^* c_{ij}^k - \sum_{r=1}^{l} u_r^* p_{ij}^r \tag{13}$$

When a unit of flow passes through an arc (i,j), the more amounts of profits and the less amounts of costs make that arc more desirable for DM. Therefore according definition of (13), a lesser λ_{ij} is more favorable; it puts the scalar in the cost category. We allocate these scalars to arcs of the network as virtual costs. The constrains (1) of model (11) guarantee that $\lambda_{ij} \geq 0$ $(\forall (i,j) \in A)$. Therefore, each arc is allocated a non-negative virtual cost instead of several costs and profits. Now, let $P \in \mathcal{P}_{st}$ (the set of all s-t paths) be an arbitrary s-t path. We define

$$\lambda(P) = \sum_{(i,j) \in A(P)} \lambda(i,j) \tag{14}$$

where $A(P)$ is the set of all arcs of P. According to above discussion, a lesser $\lambda(P)$ is more favorable for DM, therefore, we intend to find a s-t path P with minimum amount of $\lambda(P)$.

4.4. Finding a Favorable s-t Path

In (4.3), each s-t path P is associated with a virtual scalar cost $\lambda(P)$. In this section, we aim to find a shortest s-t path respect to the virtual cost as a favorable s-t path. Thus, the following shortest path problem is solved:

$$min \sum_{(i,j) \in A} \lambda_{ij} x_{ij} \tag{15}$$

$$s.t.$$

$$\sum_{j \in N^+(i)} x_{ij} - \sum_{j \in N^-(i)} x_{ji} = \begin{cases} 1 & i=s \\ 0 & i \neq s,t \\ -1 & i=t \end{cases}$$

$$x_{ij} \in \{0,1\} \quad (i,j) \in A$$

where x_{ij} $(i,j) \in A$ are the decision variables and λ_{ij} $(i,j) \in A$ are the cost parameters. It is evident that each shortest path from the source node s to the destination node t corresponds to an optimal solution to (15) [1].

As mentioned in (4.3), the virtual scalar cost allocated to each arc is non-negative so one can use Dijkstra's algorithm for obtaining a shortest s-t path. Theorem 3 guaranties that the obtained s-t path is non-dominated which is a worthwhile point.

Theorem 3. *A s-t path P with minimum $\lambda(P)$ is non-dominated.*

Proof: *The batch constraints (6) of (11) guarantees that the components of the generated CSW are positive, so the remained proof is easy.*

5. Numerical examples

In this section, we apply our method on two numerical examples and investigate the results. The first example is relatively small because small-sized problems allow the details to be thoroughly discussed.

Example 1. Consider $N = (G, \boldsymbol{c}, \boldsymbol{p}, \boldsymbol{s}, \boldsymbol{t})$ illustrated in Figure 1. G=(V,A) is a directed graph where V is the set of nodes and A is the set of arcs (V = {1,2,..,13} and |A| = 19). In the network for passing of each unit of flow from each arc (i,j), there are two types of expense, c_{ij}^1 and c_{ij}^2, and one type of profit, p_{ij}^1. The source node, s, is set to node labeled by 1 and the destination node, t, is set to node labeled by 13.

Table 1 demonstrates all the s-t paths of the network, values of costs, values of profits and state of each path including weather it is dominated and paths dominate it. We denote set of dominant paths on path P with D(P). We put some weight restrictions for the example and afterwards generate the CSW in each case (Table 2). In each case, using generated CSW, λ_{ij} is computed for each arc (i, j) and then value of $\lambda(P)$ is determined for each s-t path p (Table 3). Table 4 shows the shortest s-t paths in each case. Also in order to do better comparison, we sort s-t paths in each case by value of $\lambda(P)$. The path with the least $\lambda(P)$ outranks others (Table 5).

In this example, coefficients are more diverse and their influence on the generated CSWs is recognizable. In each case, indicators which weigh more play a greater role in computation of λ. For example, in case 1, since weight of cost in the second type is more than it in the first type, one unit difference in the second type is more effectively than one unit difference in the first type. It is why $\lambda(P1)$ is less than $\lambda(P3)$ and consequently P1 outranks P3. Theoretically, $\lambda(P)$ for a dominated s-t path P is more than a s-t path which dominates it and therefore ranks lower. Table 5 indicates that in this method a dominated s-t path can outranks some non-dominated s-t paths, for instance P2. It does not contradict the results because mentioned relation is between a dominated s-t path and the s-t paths dominated it. For example, P2 is dominated by P1. However, this path, in some indicators, is better than some of non-dominated paths like P4 and P7. It concludes, when all the cost and profit indicators are evaluated under a CSW, a dominated s-t path can have better rank than a non-dominated s-t path which does not dominate it. Finally, for a better illustration, Figure 2 shows $\lambda(P)$ for each

s-t path, P, in each case, and Figure 3 shows the ranking scores of all s-t paths with all cases of weight restrictions.

Example 2. This example is bigger and has thirty nodes and one hundred arcs. Three cost and three profit indicators are considered to evaluate the arcs. This network can be assumed as a road network where nodes are cities and each arc (i,j) is a route between city i and city j. In this way, it can be assumed that $1^{th}, 2^{th}$ and 3th cost indicators are respectively fuel consumption, travel time and expenses that $1^{th}, 2^{th}$ and 3th profit indicators are respectively earned income, safety factor and desirability. Therefore, the purpose is finding a favorable path between city 1 (the source node) and city 23 (the destination node). Since different people may have different priorities when it comes to road trips, various cases are considered for level of importance of indicators. In each case, we determine weight restrictions and compute the correspondent CSW (Table 6). Table 7 present an obtained favorable s-t path for each case, $P^{*(i)}$ for case i. For comparison, Table 8, considering all the cases, demonstrates the virtual costs for each of the favorable s-t paths. It is worth to point out that although in each case role of indicators with more relative importance is undeniable in selecting the favorable s-t path, but one cannot ignore the other indicators. There are cases where values of these indicators nullify effects of indicators with high relative importance. Comparison of and $P^{*(2)}$ accentuate this matter. Interestingly, in case 2 importance, and therefore influence, of the first type profit is more third type cost. Here, regarding first type profit, $P^{*(1)}$ is better than $P^{*(2)}$ but not in the third type cost (Table 7). Although, preference of the first type profit to the third type cost could not prevent $P^{*(2)}$ from outranking $P^{*(1)}$.

P	Costs and profit			Non-dominancy status	
	c1(P)	c2(P)	p1(P)	non-dominated	D(P)
P1	12	11	10	√	-
P2	12	13	10	×	P1,P3
P3	11	12	10	√	-
P4	10	14	6	√	-
P5	15	13	9	√	-
P6	16	15	9	×	P1,P2,P3,P5
P7	9	20	10	√	-

Table 1: *s-t paths (s=1,t=13) and their characteristics*

	Weight restrictions	CSW
Case 1	v2=2v1 , u1=v1	(0.000100,0.000200,0.000100)
Case 2	v2=2v1	(0.000100,0.000200,0.000167)
Case 3	v2=5v1	(0.000100,0.000500,0.000367)
Case 4	v2≥v1	(0.000100,0.000100,0.000100)
Case 5	v2≥2v1	(0.000100,0.000200,0.000167)
Case 6	v2≥2v1 , 3v1≤ u1 ≤5v2	(0.000100,0.000400,0.000300)

Table 2: *The weight restrictions and generated CSWs*

s-t path	Case1	Case2	Case3	Case4	Case5	Case6
P1	0.0024	0.00173	0.00303	0.0013	0.00307	0.0026
P2	0.0028	0.00213	0.00403	0.0015	0.00347	0.0034
P3	0.0025	0.00183	0.00343	0.0013	0.00317	0.0029
P4	0.0032	0.02798	0.005798	0.0018	0.003602	0.0048
P5	0.0032	0.02597	0.004697	0.0019	0.003803	0.004
P6	0.0037	0.03097	0.005797	0.0022	0.004303	0.0049
P7	0.0039	0.00323	0.00723	0.0019	0.00357	0.0059

Table 3: $\lambda(P)$ *for s-t paths*

	P*
Case1	P1
Case2	P1
Case3	P1
Case4	P1,P3
Case5	P1
Case6	P1

Table 4: *The favorable s-t paths in each case*

s-t path	Case1	Case2	Case3	Case4	Case5	Case6
P1	1	1	1	1	1	1
P2	3	3	3	2	3	3
P3	2	2	2	1	2	2
P4	4	5	6	3	5	5
P5	4	4	4	4	6	4
P6	5	6	5	5	7	6
P7	6	7	7	4	4	7

Table 5: *The ranking scores of s-t paths based on $\lambda(P)$*

	Weight restrictions	CSW					
		v1*	v2*	v3*	u1*	u2*	u3*
Case 1	v2=2v1,u1≥v3,u2≥3u1,u2≥u3	0.004823	0.009647	0.0001	0.0001	0.565907	0.320912
Case 2	v2≥v1,u1≥3v3	0.199989	0.199989	0.0001	0.0003	0.0001	5.961904
Case 3	v1≥v3,u2≥v3,u2≥u3,u3≤u1	0.002997	0.0001	0.000109	0.0001	0.821337	0.0001

Table 6: *The weight restrictions and generated CSWs*

	P^*	$c_1(P^*)$	$c_2(P^*)$	$c_3(P^*)$	$p_1(P^*)$	$p_2(P^*)$	$p_3(P^*)$	$\lambda(P^*)$
Case1	$P^{*(1)}$: 1, 5, 10, 28, 30, 23	118	43.2	72330	49100	3.1	2.3	0.816466
Case2	$P^{*(2)}$: 1,18,17, 23	72	26.8	48878	28000	2.2	1.4	7.899856
Case3	$P^{*(3)}$: 1, 5, 10, 28, 30, 23	118	43.2	72330	49100	3.1	2.3	0.780849

Table 7: *The favorable s-t paths in each case (s=1,t=23)*

P^*	case 1	case 2	case 3
$P^{*(1)}=P^{*(3)}$	0.816466	11.02858	0.780849
$P^{*(2)}$	0.999329	7.899856	0.935899

Table 8: *Virtual costs for the favorable s-t paths in each case*

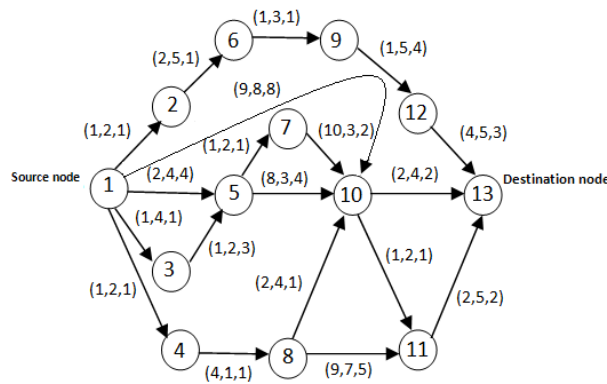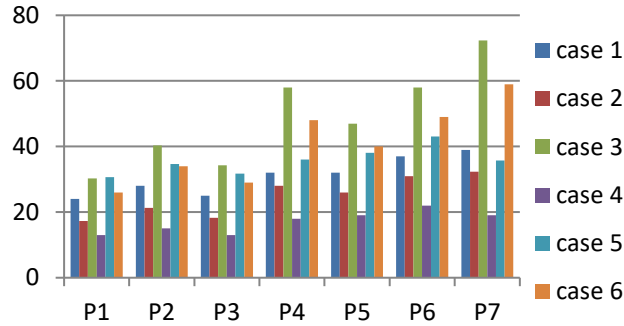**Figure 1**: *A network with 13 nodes and 19 arcs*

Figure 2: $\lambda(P)$ *for each s-t path, P, in each case. (in* 10^{-4} *scale)*
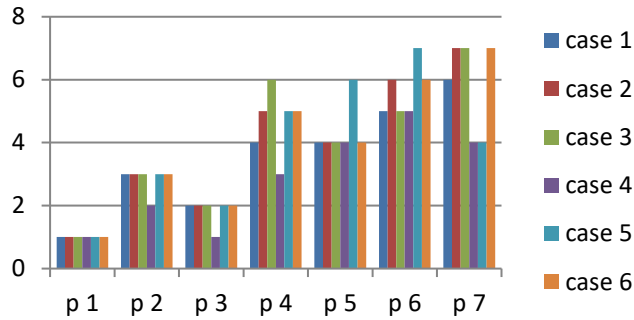

Figure 3: *The ranking scores of all s-t paths with 6 different cases of weight restrictions*

6. Conclusion

Heretofore, there are some researches using data envelopment analysis to solve multi-objective shortest path problem. Considering some problems in the previous works, we suggest a new approach using DEA to solve the problem in a more logical way. To this aim, we utilize common weights methodology which is considerable from computational and fairness aspects. Determining a non-dominated path is the main object in our mind. The DEA model serve well to carry through this purpose. Also, DM's preferences, if there are any, are considered to obtain a favorable path among the non-dominated ones from source node to destination node. Furthermore, we can use some polynomial-time shortest path algorithms, like Dijkstra, in this approach.

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