Univariate Weibull Distributions and Their Applications

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Abstract

The aim of the paper is to bring out the short and concise review of the Univariate Weibull distributions along with their properties. The area of applications is emphasized at the end of the sections.

Keywords: univariate Weibull distribution, multivariate Weibull distribution, application areas, p.d.f, c.d.f, maximum likelihood estimation
JEL classification: C4, C5

Introduction

During the years, people were conducting numerous researches in various fields connected with Weibull distributions in order to establish proper applications in the areas such as finance, insurance, economics, biostatistics, and etc.

Recently, many researchers used Weibull distributions in modeling financial data as well as in survival analysis. This article covers the review of the classical Weibull distribution following by the univariate Weibull distribution and its impact on financial modeling. Eight different models were considered; almost all supported with p.d.f's, c.d.f's and application part.

Just for the purpose of simplicity, the distributions with Fernandez and Steel approach of introducing skewness are named "Asymmetric", while the distributions with Azzalini approach of introducing skewness are named "Skewed".

Properties of the classical Weibull distribution are reviewed at the beginning of the article pointing out some important facts connected with stability and limiting properties. Due to extensive use of this distribution in various areas, discovering the original sources of the information is almost impossible. Therefore, the main purpose of this work is to provide the concise resource of the most important properties of the distribution placed within one article.

Classical Weibull Distribution

Weibull distribution can be defined as a power of the exponential distribution. If the random variable E has the standard exponential distribution with density $f_{E(x)} = e^{-x}$, x > 0, Weibull distribution can be defined as a power transformation of the exponential distribution. Therefore, for $\alpha > 0$, the variable $Y = E^{\frac{1}{\alpha}}$ has the standard Weibull distribution. With additional location and a scale parameter, we obtain a three - parameter Weibull distribution corresponding to random variable: $X = \sigma Y + \xi = \sigma E^{\frac{1}{\alpha}} + \xi$. The three - parameter Weibull probability density function, denoted by $W_\alpha(\xi, \sigma)$, takes the form.

$f_X(x) = \frac{\alpha}{\sigma}\,(\frac{x-\xi}{\sigma})^{\alpha-1}e^{-(\frac{x-\xi}{\sigma})^{\alpha}}$, $x > \xi$.
where $\alpha > 0$ is the shape parameter, σ > 0 is the scale parameter and ξ ∈ R is the location parameter. The corresponding cumulative distribution function is:
$F_X(x) = 1 - e^{-(\frac{x-\xi}{\sigma})^{\alpha}}$, $x > \xi$.

The exponential distribution is obtained for $\alpha = 1$ and the Rayleigh distribution for $\alpha = 2$ as special cases. When α is known, the maximum likelihood estimator for σ leads to the expression $(\frac{1}{n}\sum_1^n x_i^{\propto})^{\frac{1}{\propto}}$. If this quantity is substituted into the log-likelihood function, the following function of α is obtained:

$$h(\propto) = n(\log \propto - \log\left(\frac{1}{n}\sum_1^n x_i^{\propto}\right) - 1 + (\propto - 1)\frac{1}{n}\sum_1^n log x_i).$$

This function has to be maximized (numerically) with respect to α. Numerous researchers worked on this problem, Rockette at al. (1974), Conhen (1965), Harter and Moore (1968), McCool (1970), Pike (1966).Many authors following Brettoni (1964) used this distribution in reliability and quality work. It is a flexible class of distributions whose hazard function can be decrasing, constant or increasing depending on the shape parameter α. Skew models for currency exchange rates have been studied by many researchers in recent years. Westerfield (1977), Mc Farland et al. (1982), Koedjik et al. (1990), and Nolan (2001) used stable Paretian distribution, Boothe and Glassman (1987) and Tucker and Pond (1988) studied mixture of normals. Recently, asymmetric Laplace distribution was introduced by Kozubowski and Podgorski (2001), followed by Ayebo and Kozubowski (2003), who examined the goodness of fit of exponential power model. Still there is no general consensus regarding the best theoretical model, even though Chenayo et al. (1996). found the fit of a double Weibull model to be adequate.

Balakrishnan and Kocherlakota Double (Symmetric) Weibull Distribution $DW_{\alpha}\,(\xi\,,\sigma)$

Symmetrization of the classical Weibull distribution was introduced by Balakrishnan and Kocherlakota (1985) who studied order statistics and linear estimation of parameters from this distribution (see also, Dattareya Rao and Narasimhan (1989)). The process of symmetrization leads to density and distribution function of the symmetric (double) Weibull distribution given by:

$f_X(x) = \frac{1}{2} \propto |x|^{\propto-1}e^{-|x|^{\propto}}, \propto > 0, x \in R$, and $F(x) = \begin{cases} 1 - \frac{1}{2}e^{-x^{\propto}}, & x > 0 \\ \frac{1}{2}e^{-(-x)^{\propto}}, & x < 0. \end{cases}$

With the addition of location and scale parameters, a three parameter double Weibull distribution DW_{α} (ξ,σ) is obtained.

$$f_X(x) = \frac{1}{2}\frac{\alpha}{\sigma}\left|\frac{x-\xi}{\sigma}\right|^{\alpha-1} e^{-\left|\frac{x-\xi}{\sigma}\right|^{\alpha}}, x \in R.$$

If location parameter ξ = 0, this distribution is unimodal with the mode at ξ if $\alpha \le 1$; the value of the density at the mode is infinite when α is less than 1 and equal to $\frac{1}{2\sigma}$ if $\alpha = 1$. When α is greater than 1 we obtain two modes located symmetrically on each side of ξ.

Two stability properties of the double Weibull distribution are mentioned at the end. The first one is stated that $p^{\frac{1}{2}}(X_1^{\propto} + X_2^{\propto} + \cdots .. + X_{N_p}^{\propto})$ has the same distribution as

$X_1^{\propto}$, (see Kotz at al. (2001)) where $X_1, X_{2,\ldots\ldots}, X_n$ are i.i.d. random variables with the $DW_\alpha\,(0,1)$ distribution, and N_p is a geometric random variable, with the probability mass function
$P(N_p = k) = (1-p)^{k-1}p,\ k = 1, 2, 3 \ldots .$

The other property is related to minimum: $n^{\frac{1}{2}}\min(|X_1|, |X_2|, \ldots . |X_n|)$ has the same distribution as $|X_1|$ where $X_1, X_{2,\ldots\ldots}, X_n$ are random variables with $DW_\alpha\,(0,1)$, (see Mitnik and Rachev (1993)).

Kozubowski -Juric's Skew Double Weibull Distribution

Skewness is introduced into the symmetric (double) Weibull distribution previously introduced by Balakrihnan and Kocherlakota, in a two different ways. The first one follows the approach of Fernandez and Steel (1998) resulting in the distribution called "Asymmetric Double Weibull Distribution of type I", while the other model is called "Asymmetric Double Weibull Distribution of Type II".

Asymmetric double Weibull distribution of type I, $ADW_\alpha\,(\sigma, \kappa)$.

The first method consists of converting the symmetric (double) Weibull distribuion into the skew one by introducing an inverse scale factor (see Fernandez and Steel (1998)).

Such a distribution is called "Asymmetric Double Weibull distribution of Type I" and is denoted as $ADW_\alpha\,(\sigma, \kappa)$ where $\alpha > 0$ is a shape parameter, $\sigma > 0$ is a scale parameter, while $\kappa > 0$ is an inverse scale factor. It arises from the following steps:
Start with a standard exponential random variable E with density: $f_{E(x)} = e^{-x}$, x > 0.
Convert E into a standard classical Weibull random variable $W = E^{\frac{1}{\alpha}}$

Convert the density of W into a double Weibull density function. It follows

$$h(x) = \frac{1}{2}\, f(|x|), x \in R$$

Finally taking into account the procedure of Fernandez and Steel (1998) of introducing the inverse scale factor κ > 0, in addition with scale parameter σ > 0 the following p.d.f and c.d.f are obtained:

$$f(x) = \frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\begin{cases}(\kappa\, x)^{\alpha-1}e^{-(\frac{\kappa\, x}{\sigma})^{\propto}}, & x > 0\\ (-\frac{x}{\kappa})^{\alpha-1}e^{-(\frac{x}{\sigma\kappa})^{\propto}}, & x < 0\end{cases} \quad \text{and} \quad F(x) = \begin{cases}1 - \frac{1}{1+\kappa^2}\, e^{-(\frac{\kappa\, x}{\sigma})^{\propto}}, & x > 0\\ \frac{\kappa^2}{1+\kappa^2} e^{-(\frac{x}{\sigma\kappa})^{\propto}}, & x < 0.\end{cases}$$

The estimation procedure will be focused on maximum likelihood estimation. If σ is the only unknown parameter, then it can be estimated in the closed form under$ADW_\alpha\,(\sigma, \kappa)$. If both σ and κ are assumed unknown, while $\propto > 0$ is given, numerical search is needed.

Asymmetric double Weibull distribution of type II, $ADW_{\propto}^{*}(\sigma, \kappa)$

The second method includes a symmetrization of the exponential distribution (Laplace) converting it into the skew one and taking powers in order to obtain an asymmetric double Weibull model called "Asymmetric Double Weibull distribution of Type II". The steps are as follows:

Start with a standard exponential random variable E with density: $f_{E(x)} = e^{-x}$, x > 0.

Symmetrize f to obtain a double exponential (Laplace)density function:

$$h(x) = \frac{1}{2} e^{-|x|}, x \in R.$$

Introduce a skewness into h following Fernadez - Steel approach to obtain skew Laplace distribution with the following p.d.f.:

$$f(x) = \frac{1}{\sigma} \frac{\kappa}{1+\kappa^2} \begin{cases} e^{-\frac{\kappa x}{\sigma}}, & x > 0 \\ e^{-\frac{x}{\sigma\kappa}}, & x < 0, \end{cases}$$

see Kozubowsi and Podgorski (2000). An additional scale parameter σ > 0 was included.

Asymmetric double Weibull random variable is obtained by taking the power:

$$Y = \begin{cases} X^{\frac{1}{\alpha},} & x \geq 0 \\ -|X|^{\frac{1}{\alpha}}, & x < 0. \end{cases}$$

Using the c.d.f $F_X(x)$ of the asymmetric Laplace variable (see, Kozubowki and Podgorski, 2000), the cumulative distribution function $G_Y(x)$ of the Weibull random variable *Y* along with the corresponding p.d.f function is as follows:

$$G_Y(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2} e^{-\frac{k}{\sigma} x^{\alpha}}, & x > 0 \\ \frac{\kappa^2}{1+\kappa^2} e^{-\frac{1}{\sigma\kappa}(-x)^{\alpha}}, & x < 0. \end{cases} \quad \text{and} \quad g_Y(x) = \frac{1}{\sigma} \frac{\alpha\kappa}{1+\kappa^2} \begin{cases} x^{\alpha-1} e^{-\frac{k}{\sigma} x^{\alpha}}, & x > 0 \\ (-x)^{\alpha-1} e^{-\frac{1}{\sigma\kappa}(-x)^{\alpha}}, & x < 0 \end{cases}$$

The Asymmetric double Weibull model of Type II, $ADW_{\alpha}^{*}(\sigma, \kappa)$ has been fitted to currency exchange rate data set, previously considered by Nolan (2001). Kozubowski and Podgorski (2001) and Ayebo and Kozubowski (2003) worked and compared the fit to that of normal (Gaussian), asymmetric (skewed) Laplace (AL), and exponential power distribution (EP). The data set contains daily currency exchange rates, quoted in U.K. pounds, for fifteen currencies covering the period from Jan 2, 1980 to May 21, 1996. Logarithm of the price ratio for two consecutive days was compared for 42275 values of each currency. Since the likelihood function of asymmetric double Weibull distribution takes logarithm of the data values, zeros have been excluded from the sample. Thus, the conditional distribution of the changes, given that a non-zero change occurred has been modeled. To compare the fit of different models, the Kolmogorov - Smirnov (K-S) distance is computed between the data and the model distributions using the data set without zeros. Computed values of K-S distance showed the asymmetric double Weibull model of Type II to be the best.

Flaih Elsalloukh Mendi Milanova's Asymmetric Double Inverted Weibull Distribution, ADIW $(\beta, \alpha, \epsilon, \mu)$

Following the same Fernandez and Steel approach of introducing skewness into inverting Weibull distribution, the four- parameter Asymmetric Double Inverted

Weibul,l $ADIW$ (β, α, ϵ, μ) distribution is introduced along with its probability density functiona and the corresponding distribution function.

$$f(x) = \begin{cases} \frac{\beta}{2\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-(\beta+1)} e^{-\frac{1}{\alpha}(\frac{x-\mu}{1+\epsilon})^{-\beta}}, & x \geq \mu \\ \frac{\beta}{2\alpha}(\frac{\mu-x}{1-\epsilon})^{-(\beta+1)} e^{-\frac{1}{\alpha}(\frac{\mu-x}{1-\epsilon})^{-\beta}}, & x < \mu \end{cases} \quad \text{and} \quad F(x) = \begin{cases} 1 - \frac{1+\epsilon}{2}\left[1 - e^{-\frac{1}{\alpha}(\frac{x-\mu}{1+\epsilon})^{-\beta}}\right], & x \geq \mu \\ \frac{1-\epsilon}{2}\left[1 - e^{-\frac{1}{\alpha}(\frac{\mu-x}{1-\epsilon})^{-\beta}}\right], & x < \mu. \end{cases}$$

where $\propto > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, μ is a location parameter, while $-1 < \epsilon < 1$ is the skewness parameter.

Application

For the purpose of applications, two data sets were used. In the first one, the number of million revolutions before failure for 23 endurance of deep- groove ball bearings, and in the second one, the time intervals, in hours, between failures of the air conditioning system of an airplane were considered. The MLE's of the unknown parameters and log likelihood of four distributions are compared: *Gamma*(λ,α), *Weibull*(λ,α), *Exponentiated Exponetial* (λ,α) and $ADIW$ (β, α, ϵ, μ). While the first model did not show a good fit according in terms of negative log likelihood values, the second one proved to be the best outperforming other competitors. It can be concluded that $ADIW$ (β, α, ϵ, μ) is very sensitive to the type of data but at least in some cases the $ADIW$ (β, α, ϵ, μ) works better than Gamma, Weibull or Exponential .

Flaih - Elsalloukh - Mendi - Milanova's Exponentiated Inverted Weibull Distribution, $\mathbf{EIW(\beta, \theta)}$

The authors proposed the extension of the standard inverted Weibull distribution to the standard exponentiated inverted Weibull distribution by adding one more shape parameter. The distribution function of the standard exponentiated inverted Weibull distribution $EIW(\beta, \theta)$along with the density takes the form:

$$F_\theta(x) = (e^{-x^{-\beta}})^\theta,\ x, \beta, \theta > 0\ , \quad f_\theta(x) = \theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^\theta,\ x > 0.$$

Application

Two distribution $IW(\beta, \theta)$- inverted Weibull distribution and $EIW(\beta, \theta)$ exponentiated inverted Weibull distribution are fitted into the uncensored data set consisted of 100 observations concerning tensile strength of carbon fibers. Shape parameters θ and β were estimated along with the values of the log likelihood functions. The log likelihood ratio test has been performed, provided a significantly better fit for the exponentiated inverted Weibull distribution.

< - $\frac{1}{\alpha}$ > power of Asymmetric Laplace, $\mathbf{ADIW^*_\propto(\sigma\ , \kappa)}$

Knowing that random variable *X* has Asymmetric Laplace distribution, X ~ AL, it can be proved that the random variable $X^{<-\frac{1}{\alpha}->}$ follows a new distribution called Asymmetric Double Inverted Weibull distributin, assigned with $ADIW^*_\propto$(σ, κ). Applying Fernandez and Steel approach of introducing skewness, the c.d.f and the corresponding p.d.f of this model can be calculated as:

$$F(x) = \begin{cases} \frac{\kappa^2}{1+\kappa^2}\left[1 - e^{-\frac{1}{\kappa\sigma(-x)^\alpha}}\right], & x < 0 \\ \frac{\kappa^2}{1+\kappa^2} + \frac{1}{1+\kappa^2} e^{-\frac{\kappa}{\sigma(-x)^\alpha}}, & x \geq 0 \end{cases}$$ and $$f(x) = \frac{\alpha}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases} \frac{1}{(-x)^{\alpha+1}} e^{-\frac{1}{\kappa\sigma(-x)^\alpha}}, & x < 0 \\ \frac{1}{x^{\alpha+1}} e^{-\frac{\kappa}{\sigma x^\alpha}}, & x \geq 0. \end{cases}$$

Ali - Woo's Skew -Symmetric Reflected Distributions, $\mathbf{SDW_\propto(\sigma, c)}$

Recalling the procedure of obtaining density and the corresponding distribution function of the Double (Symmetric) Weibull distribution described in the Section 3, using the *sign*(x) notation, the c.d.f of DW_α (ξ, σ) can be reformulated into the form:

$$F(x) = \frac{1}{2} + \frac{1}{2} sign(x)\left(1 - e^{-|x|^\alpha}\right).$$

Along with the previous equation, this time using the Azzalini's approach (2005) of introducing skewness in the symmetric distribution and adding the scale parameter σ, the p.d.f of the Skew Symmetric Reflected Weibull Distribution $SDW_\propto(\sigma, c)$ takes the form:

$$f(z, c) = \frac{1}{2}\frac{\alpha}{\sigma}\left|\frac{z}{\sigma}\right|^{\alpha-1} e^{-\left|\frac{z}{\sigma}\right|^\alpha}\left[1 + sign(\frac{cz}{\sigma})(1 - e^{-\left|\frac{cz}{\sigma}\right|^\alpha})\right], z \in R$$

The elementary calculation for c.d.f leads to:

$$F(z, c) = \begin{cases} 1 - e^{-z^\alpha} + \frac{1}{2(1 + c^\alpha)} e^{-(1+c^\alpha)z^\alpha}, z \geq 0, c \geq 0 \\ \frac{1}{2(1 + c^\alpha)} e^{-(1+c^\alpha)(-z)^\alpha}, \quad z < 0, c > 0. \end{cases}$$

Furthermore, if a random variable X has Skew Laplace Distribution, thus, $X \sim (\lambda = c^\alpha)$, λ >0, then $X^{<\frac{1}{\alpha}>}$ has Skew Symmetric Reflected Distribution $SDW(1, c)$. Thus, a very interesting relationship between Ali- Woo model and the Skew Laplace distribution can be obtained. It is proved that applying the Azzalini process of obtaining skewness, the Skew Laplace distribution can be written in the form

$$f(x) = \frac{1}{2}e^{-|x|}\left[1 + sign(\lambda x)\left(1 - e^{-|\lambda x|}\right)\right]$$

Calculating the c.d.f for $Y = X^{<\frac{1}{\alpha}>}$, $y < 0$, and inserting the corresponding parameters into above equation, the following model is easily formed:

$$g(y) = \frac{1}{2}\alpha|y|^{\alpha-1}e^{-|y|^\alpha}e^{-|\lambda||y|^\alpha}$$

It can be seen that above equation is Ali- Woo Skew-Symmetric Reflected Weibull model, $SDW(1, \lambda)$ for $\lambda = c^\alpha$. The similar procedure for $y > 0$ leading to:

$$g(y) = \frac{1}{2}\alpha y^{\alpha-1}e^{-y^\alpha}(2 - e^{-|\lambda|y^\alpha}).$$

Ali - Woo Nadarajah's Skew -Symmetric (Double) Reflected (Inverted)Weibul Distribution $\mathbf{SDIW(\sigma, c)}$

The last model is obtained by applying the Azzalini model of introducing skewness into the Double Inverted Weibull Distribution $DIW_\propto(\sigma, c)$. The model is presented with formulas for p.d.f, c.d.f. but no further connections with the previous models could be found.

The intention was to prove the connection between the random variable X following the Skew Laplace distribution, X ~ SL(λ), λ > 0, and $Y = \frac{1}{X^{<\frac{1}{\alpha}>}}$ having $SDIW_{\propto}(1, c)$ distribution. Comparison gave no results regarding this possibility.
In order to present the model we just calculated the probability density function which takes the form:

$$f(x) = \frac{\alpha}{2\sigma} \frac{1}{\left|\frac{x}{\sigma}\right|^{\alpha+1}} e^{-\frac{1}{\left|\frac{x}{\sigma}\right|^{\alpha}}} \left[1 + sign(-c\frac{x}{\sigma})e^{-\frac{1}{\left|\frac{x}{\sigma}\right|^{\propto} c^{\propto}}}\right].$$

Conclusion

Eight different variation of, both symmetric and asymmetric (skewed) univariate Weibull distributions have been considered along with some of their properties. We hope that this work will help other researchers in gathering additional theoretical concepts as well as finding suitable applications that will prove the usefulness of univariate Weibull distributions in numerous areas.

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