Duality between the Short Run Profit and Production Function

Ilko Vrankić
Faculty of Economics and Business - Zagreb, Croatia
Mira Krpan
Faculty of Economics and Business - Zagreb, Croatia

Abstract
Duality in microeconomic theory takes an important role in the theoretical work and in empirical research. The dual description of convex set implies the unique structure of microeconomic phenomena in the consumer and producer theory. In this article the unique structure of microeconomic phenomena is analysed on the problem of duality between the short run production and profit function which describe the firm's technology. The primal problem includes short run profit maximisation, in which the choice variable is the quantity of labour and the normalised short run profit function is derived. The dual model includes minimising the sum of variable costs and maximum profit. Special attention is given to the analysis of the first order necessary condition of both optimisation problems which imply duality between the Hotelling's lemma that generates the labour demand function in the short run and its dual which generates the inverse demand function for labour. It is shown how the dual of Hotelling's lemma generates the solution of the primal problem, and how the Hotelling's lemma generates the solution to the dual problem. Finally, the comparative static results that accompany the short-run production theory, important from an empirical standpoint, are especially emphasised.

Keywords: duality research, microeconomic theory, short run profit function, market, production function, Hotelling's lemma, comparative static analysis.
JEL classification: D00

Introduction
Duality in microeconomic theory includes derivation and recovering of the alternative representations of consumer preferences and production technology (Blume, 2008; Diewert, 1982; Shepard, 1970). In this paper duality between the short run production and profit functions is analysed. From the Hotelling's time, when duality in production was first analysed, McFadden (1966) first proved McFadden duality theorem between the profit and production function (McFadden, 1978). An alternative proof was given by Lau (1969c) who defined the normalised profit function (Lau, 1978), which is profit expressed in the output units. The following years were marked with proving duality with different regularity conditions and with empirical estimation of the technology parameters starting from the profit function (Arnade, 2007; Gao, 2008).
The subject of our analysis are two optimization models, short run profit maximisation model, in which the choice variable is the quantity of labour and the normalised short run profit function is derived, and the dual model which includes minimising the sum of variable costs and maximum profit, and the production function is derived. Our goal is to show that when variables in one model become parameters in the other, the dual model is obtained and vice versa. It will also be shown that both

models generate the same system of labour demand functions and its inverse demand labour function. Finally, emphasis is given to the comparative statics analysis. What gives the original scientific contribution of the paper next to overall analytical duality analysis is its graphical analysis. The results are illustrated with a numerical example.

From the production function to the normalised profit function

It is assumed that the firm is perfectly competitive in the output and input market. Technology in the short run is characterised by the production function $y = f(L, \overline{K})$, where y is the output quantity, L is the quantity of the variable input labour and $\overline{K}$ is the quantity of the fixed input capital. In the short run the perfectly competitive firm chooses the profit maximising labour and output quantities.

Since the optimal variable input and output quantities are not influenced by the quantity of the fixed input, the short run profit function will be defined below as the difference between the total revenue and variable cost,

$$\pi(p, w, \overline{K}) = \max_{L} pf(L, K) - wL\,, \qquad (1)$$

where p is the product price and w is the price of the variable input. By dividing all prices in the model by the product price and expressing them in the units of product, the upper model reduces to the following equivalent model:

$$\frac{\pi}{p}(\frac{w}{p}, \overline{K}) = \max_{L} f(L, \overline{K}) - \frac{w}{p}L\,. \qquad (2)$$

The optimal value function in this optimisation model is called the normalised profit function after Jorgenson and Lau (Lau, 1978). It is the function that gives the maximum profit in the short run expressed in the units of output. The firm chooses the quantity of variable input taking into account the given product price, the quantity of the fixed input and the variable input price. Therefore, the solution of the above optimisation problem includes the variable input demand function, the supply function and the maximum short run normalised profit function, all depending on the variable input price and the fixed input quantity.

Below the graphical analysis is given. We start from the production curve that represents technology in the short run and gives maximum output quantity that can be produced given the fixed input quantity and the given technology. It is assumed that the production function is differentiable on its domain which implies that the production curve has the unique tangent in each point. We also assume that the production function is concave and that the production process in the short run is characterised by the diminishing marginal product of labour.

Let's start from the chosen quantity of labour L^0 that together with fixed capital gives $y^0 = f(L^0, \overline{K})$ and look at the point (L^0, y^0) on the production curve. Now let's draw a tangent on the production curve at this point. The slope of the tangent at the point is the value of the marginal product of labour at this point, $\frac{\partial f(L)}{\partial L}(L^0) = f_L(L^0)$, where the equation of the tangent with the given slope and the point is

$$y = y^0 + f_L(L^0)(L - L^0)\,. \qquad (3)$$

We can look at the tangent from another perspective. The production curve can be interpreted as the real revenue curve, where real revenue is obtained by dividing total revenue with the product price. It is actually the revenue expressed in the units of output. Next we add the graph of the real variable costs whose equation is $y = \frac{w}{p} L$. In other words costs are expressed in the units of output. It is the line with slope equal to the real wage. Graphically we are looking for the labour quantity which gives the biggest difference between the real revenue and the real variable costs. After Lau we call this difference the normalised profit, or profit expressed in the units of output,

$$\frac{\pi}{p}(\frac{w}{p}, \bar{K}) = \max_{L} f(L, \bar{K}) - \frac{w}{p} L \,. \qquad (4)$$

The real variable cost curve is actually the isoprofit curve which gives all the combinations of labour and production that give the zero level of normalised profit. The equation of the isoprofit line is $y = \frac{\Pi}{p} + \frac{w}{p} L$, which for $\frac{\Pi}{p} = 0$ collapses to $y = \frac{w}{p} L$. The normalised profit is graphically represented as the intercept of the isoprofit line. Since our interest is in finding the maximum profit, we'll move isoprofit lines up until the tangency of the production curve and the isoprofit line is reached. For this level of labour isoprofit line is the tangent on the production curve and the real wage is equal to the marginal product of labour. This result is also contained in the first order necessary condition for the problem of normalised profit maximisation,

$$\left(\frac{w}{p}\right)^0 = \frac{\partial f(L, \bar{K})}{\partial L} \,, \qquad (5)$$

which is obtained by differentiating (4) with respect to *L*. The equation of the isoprofit line representing maximum profit is

$$y = \left(\frac{\Pi}{p}\right)^* + \left(\frac{w}{p}\right)^0 L \,. \qquad (6)$$

The second order sufficient conditions imply decreasing marginal product of labour (Mas-Colell, 1995),

$$f_{LL} < 0.$$

Let's assume now that the real wage increases, $\left(\frac{w}{p}\right)^1 > \left(\frac{w}{p}\right)^0$. For the given technology and fixed factor of production, the active producer will adjust to the new change and hire less labour at new real wage. This will lead to decreased production. Just for an illustration, let's analyse new isoprofit line

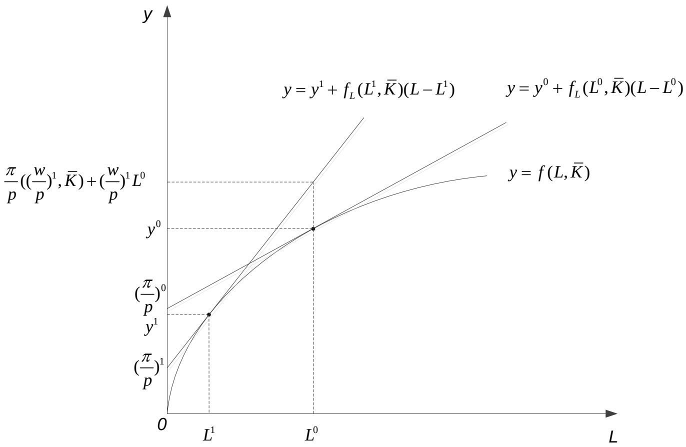
$$y = \frac{\Pi}{p} + \left(\frac{w}{p}\right)^1 L \,, \qquad (7)$$

and look for technologically feasible profit maximizing quantity of labour. The analysis brings us to the point of tangency between the production curve and the isoprofit line. By changing real wage, other tangents are obtained and the production curve is enveloped by tangents. Intercepts of tangents on the vertical axis represent the value of maximum normalised profit for various values of real wages and for the given level of the capital. At the real wage $\left(\frac{w}{p}\right)^0$ the optimal labour level is L^0. The equation of a tangent on the production curve at $\left(\frac{w}{p}\right)^0$ is $y=\left(\frac{w}{p}\right)^0 L+\left(\frac{\pi}{p}\right)^0$ or equivalently, the equation of a line passing through the point $\left[L^0, f(L^0)\right]$ with the given slope $\left(\frac{w}{p}\right)^0$ is $y-f(L^0)=\left(\frac{w}{p}\right)^0 (L-L^0)$. At the real wage $\left(\frac{w}{p}\right)^1$ the optimal labour level is L^1. The equation of a tangent on the production curve at $\left(\frac{w}{p}\right)^1$ is $y=\left(\frac{w}{p}\right)^1 L+\left(\frac{\pi}{p}\right)^1$ or equivalently, the equation of a line passing through the point $\left[L^1, f(L^1)\right]$ with the given slope $\left(\frac{w}{p}\right)^1$ is $y-f(L^1)=\left(\frac{w}{p}\right)^1 (L-L^1)$. Since the graph of the concave production function is below its tangent, the following inequalities for the two labour levels hold: $f(L^1)\le f(L^0)+\left(\frac{w}{p}\right)^0 (L^1-L^0)$ and $f(L^0)\le f(L^1)+\left(\frac{w}{p}\right)^1 (L^0-L^1)$. From these two previous inequalities the important comparative statics result from the production theory follows,

$$(L^0-L^1)\left[\left(\frac{w}{p}\right)^0-\left(\frac{w}{p}\right)^1\right]\le 0. \qquad (8)$$

It implies that an increase in real wage decreases demand for labour of the profit maximising firm for the given technology and fixed input.

Figure 1
Deriving the Normalised Profit Function



Source: Authors' illustration

From the normalised profit function to the production function

In the previous model, which can be called the primal problem, the real wage was given and we looked for the profit maximizing level of labour in the short run. At real wage $\frac{w^0}{p^0}$ the optimal level of labour is L^0 and the maximum normalised profit is $\pi(\frac{w^0}{p^0})$. If the real wage changes, the producer has two options. He can stay passive and employ the same level of labour or can adjust to new market changes. The behaviour of the passive producer can be described geometrically with the line $\frac{\pi}{p} = f(L^0, \overline{K}) - \frac{w}{p}L^0$ in the space where the real wage is on the horizontal axis and the profit on the vertical axis. The intercept of the line on the vertical axis is $f(L^0, \overline{K}) = y^0$. This line can be called the normalised profit function of the passive producer. If the producer is active on the other hand, he will hire profit maximising level of labour at every real wage and accomplish higher profit. By comparing the graph of the maximum normalised profit function of the active producer and the graph of the normalised profit function of the passive producer, which is the line, it can be concluded that the first graph is above its tangent, which means that the normalised profit function is convex in real wage. Since the maximum profit is always higher than the profit reached by hiring the given level of labour L^0, the following inequality holds

$$\frac{\pi}{p}(\frac{w}{p}, \overline{K}) \geq f(L^0, \overline{K}) - \frac{w}{p}L^o. \tag{9}$$

By changing places of the normalised profit function and the production function, the following inequality follows:

$$f(L^0, \overline{K}) \leq \frac{\pi}{p}(\frac{w}{p}, \overline{K}) + \frac{w}{p}L^o. \tag{10}$$

Since the level of labour L^0 is optimal at real wage $\frac{w^0}{p^0}$, it follows

$$f(L^0, \overline{K}) = \frac{\pi}{p}\left[\left(\frac{w}{p}\right)^0, \overline{K}\right] + \left(\frac{w}{p}\right)^0 L^0. \tag{11}$$

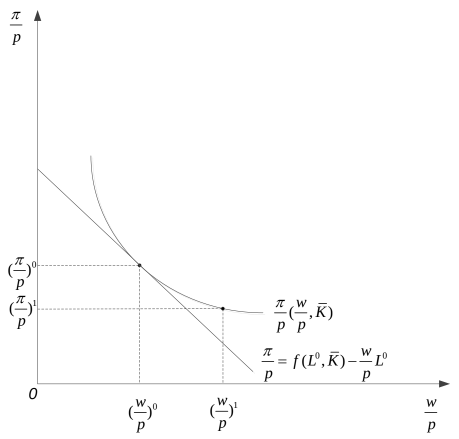
Therefore, the production at L^0 is the lower bound of the expression on the right hand side for various level of real wage and for the real wage equal to $\left(\frac{w}{p}\right)^0$, the equality is satisfied. We are looking for the real wage at which the given level of labour is hired and the word is about minimum. The choice variable is the dual variable, real wage,

$$f(L^0, \overline{K}) = \min_{\frac{w}{p}} \frac{\pi}{p}(\frac{w}{p}, \overline{K}) + \frac{w}{p}L^o \tag{12}$$

The first order necessary condition brings us to the following result:

$$L^0 = -\frac{\partial \frac{\pi}{p}(\frac{w}{p}, \overline{K})}{\partial \left(\frac{w}{p}\right)}. \tag{13}$$

Figure 2
The profit function of the active and passive producer



Source: Authors' illustration

The real wage at which the producer will hire the given level of labour L^0 is characterised by the equality between L^0 and the change in maximum profit in the short run on the small unit of real wage change. This result is in the literature called Hotelling's lemma. It implies how the demand for labour in the short run can be derived by starting from the profit function. It can be noticed that the solution to the primal optimisation problem is equal to the marginal profit, $L^0 = \frac{\partial \frac{\pi}{p}(\frac{w}{p}, \overline{K})}{\partial \left(\frac{w}{p}\right)}$, and that the solution to the dual optimisation problem is equal to the marginal product of labour, $\left(\frac{w}{p}\right)^0 = \frac{\partial f(L, \overline{K})}{\partial L}$.

Table 1
The symmetry of problem solving and the results of the models

The primal problem The dual problem	
$\frac{\pi}{p}\left[\left(\frac{w}{p}\right)^0, \overline{K}\right] = \max_L f(L, \overline{K}) - \left(\frac{w}{p}\right)^0 L$	$f(L^0, \overline{K}) = \min_{\frac{w}{p}} \frac{\pi}{p}(\frac{w}{p}, \overline{K}) + \frac{w}{p}L^o$
First order necessary conditions:	**First order necessary conditions:**
$\left(\frac{w}{p}\right)^0 = \frac{\partial f(L, \overline{K})}{\partial L}$	$L^0 = -\frac{\partial \frac{\pi}{p}(\frac{w}{p}, \overline{K})}{\partial \left(\frac{w}{p}\right)}$

Source: Authors

Example

We choose the production function $y = f(L,K) = 2L^{\frac{1}{4}}K^{\frac{1}{4}}$ which describes decreasing returns to scale. It is assumed that the short run level of capital is 16. Therefore the short run production function is $y = f(L,\overline{K}) = 4L^{\frac{1}{4}}$. The short run profit maximisation problem is described by $\frac{\pi}{p}(\frac{w}{p}, K = 16) = \max_{L} 4L^{\frac{1}{4}} - \frac{w}{p}L$. The choice variable in the above maximisation model is the level of labour in the short run. By differentiating the goal function with respect to L, the first order necessary condition is obtained $\frac{w}{p} = L^{-\frac{3}{4}} / ^{-\frac{4}{3}}$. The solution to the above equation gives the labour demand function $L(\frac{w}{p},\overline{K}) = \left(\frac{w}{p}\right)^{-\frac{4}{3}}$. By substituting the derived input demand function in the short run production function, the supply function for the perfectly competitive profit maximising firm is obtained, $y(\frac{w}{p},\overline{K}) = 4(\frac{w}{p})^{-\frac{1}{3}}$. Finally, substitution of the derived input demand function and the supply function in the goal function gives the short run normalised profit function, $\pi(\frac{w}{p},\overline{K}) = 4(\frac{w}{p})^{-\frac{1}{3}} - \frac{w}{p}\left(\frac{w}{p}\right)^{-\frac{4}{3}} = 3\left(\frac{w}{p}\right)^{-\frac{1}{3}}$.

In the dual problem the starting point is the short run normalised profit function and the goal is to recover the short run production function. According to (12), the dual optimisation problem is $f(L,\overline{K}) = \min_{\frac{w}{p}} 3\left(\frac{w}{p}\right)^{-\frac{1}{3}} + \frac{w}{p}L$. By differentiating the goal function with respect to the real wage, the first order necessary condition is obtained, $-\left(\frac{w}{p}\right)^{-\frac{4}{3}} + L = 0 \Rightarrow L = \left(\frac{w}{p}\right)^{-\frac{4}{3}}$ and $f(L,\overline{K}) = 3(L^{-\frac{3}{4}})^{-\frac{1}{3}} + L^{-\frac{3}{4}}L = 4L^{\frac{1}{4}}$. Therefore, the starting production function is recovered.

Conclusion

In this paper duality between the short run production and profit functions is analysed. In the short run profit maximisation model, in which the choice variable is the quantity of labour the normalised short run profit function is derived, and in the dual model which includes minimising the sum of variable costs and maximum profit, the production function is derived. It is shown that when variables in one model become parameters in the other, the dual model is obtained and vice versa. Analysis of the first order necessary condition of both optimisation problems imply duality between the Hotelling's lemma that generates the labour demand function in the short run and its dual which generates the inverse demand function for labour. Next to overall analytical duality analysis, the paper includes its graphical analysis by enveloping the production curve with its tangents. Future research will be devoted to empirical application of the profit function.

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About the authors

Ilko Vrankić, PhD, is an Associate Professor of Microeconomics and Advanced Microeconomics at the Department of Economic Theory, Faculty of Economics and Business, University of Zagreb. His current research areas are mathematical modelling of economic phenomena and dual approaches to the optimisation problems in microeconomic theory. He is the (co)author of number of articles in international and national journals. Author can be contacted at **ivrankic@efzg.hr.**

Mira Krpan, PhD, is an Assistant Professor of Microeconomics and Principles of Economics at the Department of Economic Theory, Faculty of Economics and Business, University of Zagreb. Her current research areas are mathematical modelling of economic phenomena and dual approaches to the optimisation problems in microeconomic theory. She is the (co)author of number of articles in international and national journals. Author can be contacted at **mira.krpan@efzg.hr.**