Two – Dimensional Modelling of Financial Data

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Abstract

The article deals with modelling of two-dimensional financial data set using Weibull distribution extended to two-dimensional setting. The generalization in two dimensions is not direct, it goes through representation of one-dimensional asymmetric Laplace distribution. The characteristics of the new distribution are described, and parameters are estimated using the method of moments. Statistical package R is used to perform numerical search. At the end, the application of this new two- dimensional family of distributions is discussed.

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Introduction

Symmetrical and non-symmetrical distributions have resulted in numerous uses in financial realm. The Weibull distribution, as one of the most common models in statistical applications, is widely used in the field of science, engineering, finance and economics, (Halinan 1993, Johnson et al., 1994). To use this model in financial application, the model needs to be extended to the entire real axis by symmetrisation of density

$$f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}}, \quad x > 0\,, \tag{1}$$

Leading to:

$$f(x) = \frac{\alpha}{2} |x|^{\alpha-1} e^{-|x|^{\alpha}}, \quad x \neq 0, \tag{2}$$

(Balakrishnan and Kocherlakota 1985).

Since, there is extensive empirical evidence that the logarithms of financial investment returns are not distributed symmetrically, it is necessary to look for families of distributions that are not symmetric. In the symmetrical Weibull distribution (2) asymmetry is introduced using Fernandez and Steel approach (see, Fernandez and Steel, 1998) that introduced inverse scale factors to convert the symmetric distribution into the asymmetric one. The new density takes the form:

$$(x) = \begin{cases} cf(x\kappa), & x \geq 0 \\ cf\left(\frac{x}{\kappa}\right), & x < 0. \end{cases} \tag{3}$$

The constant c is calculated by integrating over the interval $(-\infty. +\infty)$ resulting to

$$c = \frac{2k}{k^2 + 1}.$$

In addition, with scale parameter $\sigma > 0$, the following density is obtained:

$$f(x) = \frac{1}{\sigma^{\alpha}} \frac{\alpha\kappa}{1+\kappa^2} \begin{cases} (\kappa\, x)^{\alpha-1} e^{-(\frac{\kappa\, x}{\sigma})^{\alpha}}, & x > 0 \\ (-\frac{x}{\kappa})^{\alpha-1} e^{-(\frac{x}{\sigma\kappa})^{\alpha}}, & x < 0. \end{cases} \tag{4}$$

This produced a new flexible family of distributions with the parameters α, σ and κ suitable for evaluation of the parameters and successful modelling of financial data in one dimensional case, (Juric and Kozubowski 2005).

However, this approach does not allow a generalisation to two-dimensional setting. Therefore, the process takes place indirectly through identity of Laplace random variable Y, known to have a representation $Y = \sqrt{2E}\,Z$, where E is a standard exponential random variable independent of standard normal random variable Z, (see Kotz et al. 2001). This representation is a mixture of normal distributions, in other words a normal distribution with a stochastic variance 2E.

Moreover, Kozubowski and Podgorski (Kozubowski and Podgorski 2000) proved that variable $Y = mE + \sqrt{2E}\ Z$, has an asymmetric Laplace distribution.

Furthermore, for the symmetrical Laplace random variable L and independent stable random variable S with an index $\alpha \in (0,1)$ defined by Laplace's transform

$$g(t) = Ee^{-tS} = \int_0^{+\infty} e^{-st} f_S(s)ds = e^{-t^\alpha} \quad (5)$$

can be shown easily that the variable $Y = \frac{L}{S}$ has a symmetric Weibull distribution with the parameters α and $\sigma = 1$. This conclusion with:

$$Y = mE + \sqrt{2E}\,Z \quad (6)$$

having an asymmetric Laplace distribution lead to the idea that

$$W = \frac{mE + \sqrt{2E}\,X}{S} \quad (7)$$

where E, $X \sim N(0, \tau^2)$ and S being all independent has the asymmetric Weibull distribution with the parameters α,

$$\sigma = \tau \text{ and } \kappa = \frac{\sqrt{m^2+4\tau^2}-m}{2\tau} \quad (8)$$

(Juric, Kozubowski and Perman, 2019).
This formula can be naturally generalized to several dimensions by replacing random variable X with multivariate normal vector X.

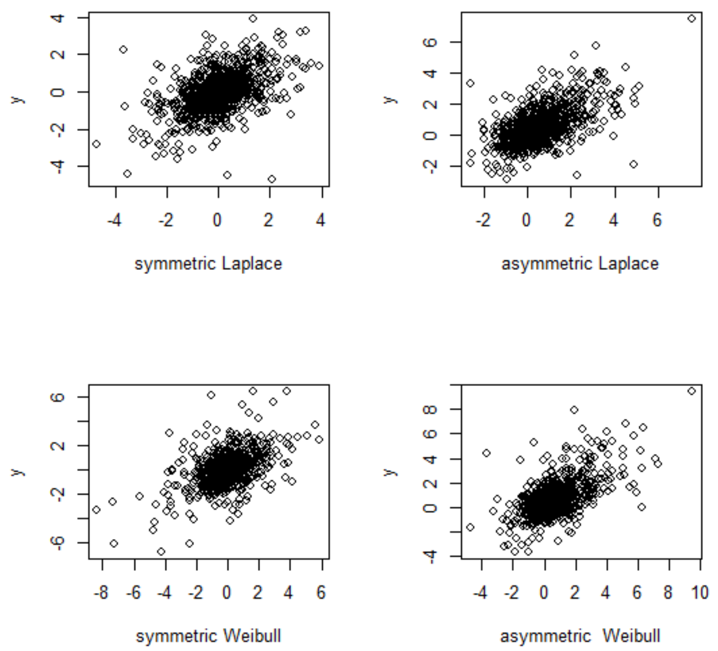
By doing so, we consciously renounce the possibility of the components being independent, but this new family still has many beneficial properties. This pathway can be an alternative to copulas, which are often used to construct distributions with given marginal distributions.

Defining $W = \frac{mE+\sqrt{2E}\,X}{S}$, a random vector W has an asymmetric multivariate Weibull distribution with $0 < \alpha \leq 1$, where $m \in R^d$, Σ is $d \times d$ positively semi-definite matrix and random vector $X \sim N_d(0, \Sigma)$ has a multivariate normal distribution with an expected value of 0 and the covariance matrix Σ, where E is a standard exponential random variable, S standard stable random variable independent of E given by the Laplace transform (5). The symmetric case is obtained for $m = 0$, while for $\alpha \in (0.1)$ the unimodal distribution is obtained, (Juric et al., 2019).

SIMULATION

Considering the previous definition of the multivariate random vector W, the simulation of bivariate asymmetric Weibull distributions is possible using the well- known algorithm of Chambers, Mellow and Stuck (1976) modified by Weron (1996). The algorithm provided the steps to generate the stable random variable with parameter α ∈ (0, 1). Figure 1 shows the simulated sample points from symmetric and asymmetric Laplace and Weibull bivariate distributions. The asymmetry can be clearly observed from the figures on the right.

Figure 1
Simulated sample points from symmetric and asymmetric Laplace and Weibull bivariate distributions



Source: Author's illustration

Literature review

Symmetric and asymmetric versions of the Weibull distributions are found to be suitable models for currency exchange rates, (see Mittnik and Rachev 1993, Chenyao *et al*. 1996. Kozubowski and Podgorski 2001, Malvergne and Sornette 2005).

Many authors focused their studying to the skew models for currency exchange rates using stable Paretian distribution (see, Westerfield 1977, Mc Farland et al. 1982, Koedjik et al. 1990, and Nolan 2001), while Boothe and Glassman (1987), Tucker and Pond (1988) applied mixture of normals.

Knowing that there are many bivariate or multivariate Weibull distributions, based on bivariate or multivariate exponential distributions that are obtained as extensions of univariate exponential distribution, Hanagal (2009) presented a new multivariate Weibull distribution with many interesting properties. The main reason Hanagal's model has been chosen is that this multivariate Weibull distribution is obtained from multivariate exponential model of Marshall-Olkin (see, Marshal- Olkin 1997) which is the Multivariate Weibull distribution having the marginals as exponentials.

The non-Gaussian properties of the distributions of the asset returns are presented in the work of authors Malevergne and Sornette (see, Malaverne and Sornette 2005). In this work the multivariate distribution for asset returns is presented where marginal distributions are parameterized in terms of modified Weibull distribution. Two key parameters, c-the exponent and χ-the characteristic scale of the modified Weibull distribution are derived. Statistical tests of this parametrization are discussed.

Hsiaw-Chan-Yeh's (2009) introduced multivariate semi-Weibull distribution proposing two more general multivariate distributions with Weibull marginals that are constructed following Marshall and Olkin results. More general cases are considered.

Many authors proposed bivariate distributions for modelling currency exchange data sets. Juric et al. (2019) modelled two exchange rate data sets using the Multivariate double Weibull distribution.

Amponsah et al. (2021) proposed a new stochastic model for bivariate episodes driven by a gamma sequence describing joint distributions of (X, N), where N is a counting variable and X is the sum of N independent gamma random variables.

Bhattacharya, Das and Kunnummal (2024) deals with the construction of bivariate distribution by combining the bivariate uniform and bivariate Weibull distributions. A 2-dimensional copula has been obtained by assuming that the marginal distribution is a two-parametric Weibull distribution. Some properties such as joint probability density function, survival (reliability) function, and hazard (failure rate) function of the modified bivariate Weibull distribution are also obtained. A combination of the maximum likelihood estimation technique and machine learning clustering algorithm is performed.

Methodology

In order to estimate parameters and produce a suitable two- dimensional model that will fit the two-dimensional financial data set, some extensive theoretical work needs to be established.

Applications of the bivariate asymmetric Weibull distribution depend on effective estimation methods. It is known that the densities of the standard stable random variable are known explicitly in special cases like $\alpha = \frac{1}{2}$ only and the maximum likelihood approach is not suitable. Therefore, the method of moments is applied.

Focus is on estimation of the parameters α, $m_{1,}$, $m_{2,}$ and the coefficients of 2×2 matrix Σ. For the simplicity, number 2 is excluded under the square root in the form $m_i \frac{E}{S} + \frac{\sqrt{E}}{S} X$.

Using the parametrization (7), the following relation can be obtained:

$$E(W_i W_j) = E\left[\left(m_i \frac{E}{S} + \frac{\sqrt{E}}{S} X_i\right)\left(m_j \frac{E}{S} + \frac{\sqrt{E}}{S} X_j\right)\right]. \tag{9}$$

Knowing that $E(E^2) = 2$, $E(S^{-2}) = \frac{1}{2}\Gamma\left(1 + \frac{2}{\alpha}\right)$ and $E(X_i) = 0$, the equation can be written in the following form:

$$E(W_i W_j) = m_i m_j \, \Gamma\left(1 + \frac{2}{\alpha}\right) + \frac{\sigma_{ij}}{2} \Gamma\left(1 + \frac{2}{\alpha}\right), \tag{10}$$

leading to

$$E(W_i{}^2) = m_i{}^2 \, \Gamma\left(1 + \frac{2}{\alpha}\right) + \frac{\sigma_{ii}}{2} \Gamma(1 + \tfrac{2}{\alpha}) \text{ and } E(W_i) = m_i \, \Gamma\left(1 + \frac{1}{\alpha}\right) \tag{11}$$

where σ_{ij} indicates $Var(X_{Ij})$.

As stated before, the estimation of the parameters will be performed by using the method of moments. The following equations are produced:

$$\overline{W_1} = \frac{1}{n}\sum_{i=1}^{n} W_{1i} \approx E(W_1) = m_1 \, \Gamma\left(1 + \frac{1}{\alpha}\right) \tag{12}$$

$$\overline{W_2} = \frac{1}{n}\sum_{i=1}^{n} W_{2i} \approx E(W_2) = m_2\,\Gamma\left(1+\frac{1}{\alpha}\right) \tag{13}$$

$$\overline{W_1^2} = \frac{1}{n}\sum_{i=1}^{n} {W_{1i}}^2 \approx E({W_1}^2) = {m_1}^2\,\Gamma\left(1+\frac{2}{\alpha}\right) + \frac{1}{2}\Gamma\left(1+\frac{2}{\alpha}\right)\sigma_{11} \tag{14}$$

$$\overline{W_2^2} = \frac{1}{n}\sum_{i=1}^{n} {W_{2i}}^2 \approx E({W_2}^2) = {m_2}^2\,\Gamma\left(1+\frac{2}{\alpha}\right) + \frac{1}{2}\Gamma\left(1+\frac{2}{\alpha}\right)\sigma_{22} \tag{15}$$

$$\overline{W_1W_2} = \frac{1}{n}\sum_{i=1}^{n} W_{1i}\,W_{2i} \approx E(W_1W_2) = m_1m_2\,\Gamma\left(1+\frac{2}{\alpha}\right) + \frac{1}{2}\Gamma\left(1+\frac{2}{\alpha}\right)\sigma_{12} \tag{16}$$

Furthermore,

$$\frac{1}{n}\sum_{i=1}^{n} {W_{1i}}^2\,{W_{2i}}^2 \approx E({W_1}^2{W_2}^2) = \Gamma\left(1+\frac{4}{\alpha}\right)\left[{m_1}^2{m_2}^2 + \frac{1}{4}({m_1}^2\sigma_{22} + {m_2}^2\sigma_{11}) + \frac{1}{6}{\sigma_{12}}^2 + \frac{1}{12}\sigma_{11}\sigma_{22} + m_1m_2\sigma_{12}\right] \tag{17}$$

The parameters m_1, m_2 will be easily estimated from (12) and (13) yielding to:

$$m_1 = \frac{\overline{W_1}}{\Gamma\left(1+\frac{1}{\alpha}\right)} \text{ and } m_2 = \frac{\overline{W_2}}{\Gamma\left(1+\frac{1}{\alpha}\right)}. \tag{18}$$

Substituting these formulas into (14) to (16) the following expressions for σ_{11}, σ_{12}, σ_{22} are obtained:

$$\widehat{\sigma_{11}} = \frac{2\left[\overline{W_1^2} - G_\alpha(\overline{W_1})^2\right]}{\Gamma\left(1+\frac{2}{\alpha}\right)} \tag{19}$$

$$\widehat{\sigma_{22}} = \frac{2\left[\overline{W_2^2} - G_\alpha(\overline{W_2})^2\right]}{\Gamma\left(1+\frac{2}{\alpha}\right)} \tag{20}$$

$$\widehat{\sigma_{12}} = \frac{2\left[\overline{W_1W_2} - G_\alpha(\overline{W}_1\overline{W}_2)\right]}{\Gamma\left(1+\frac{2}{\alpha}\right)} \tag{21}$$

$$\text{where } G_\alpha = \frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma^2\left(1+\frac{1}{\alpha}\right)}. \tag{22}$$

To estimate parameter α, all these equations are substituted in (16), so the following formula is constructed:

$$E({W_1}^2{W_2}^2) = \Gamma\left(1+\frac{4}{\alpha}\right)\cdot$$

$$\left\{\frac{(\overline{W}_1\overline{W}_2)^2}{\Gamma^4\left(1+\frac{1}{\alpha}\right)} + \frac{1}{2}\frac{\overline{W_1}^2\left[\overline{W_2^2} - G_\alpha(\overline{W_2})^2\right] + \overline{W_2}^2\left[\overline{W_1^2} - G_\alpha(\overline{W_1})^2\right]}{\Gamma^2\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right)} + \frac{2}{3}\frac{\left[\overline{W_1W_2} - G_\alpha(\overline{W}_1\overline{W}_2)\right]^2}{\Gamma^2\left(1+\frac{2}{\alpha}\right)} + \frac{1}{3}\frac{\left[\overline{W_1^2} - G_\alpha(\overline{W_1})^2\right]\left[\overline{W_2^2} - G_\alpha(\overline{W_2})^2\right]}{\Gamma^2\left(1+\frac{2}{\alpha}\right)} + \frac{2\overline{W}_1\overline{W}_2\left[\overline{W_1W_2} - G_\alpha(\overline{W}_1\overline{W}_2)\right]}{\Gamma^2\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right)}\right\}$$

Since the formula depends only on the parameter statistical package R is used to obtain the value of α.

Results

To show the modelling potential of the two dimensional multivariate Weibull distribution, the data set was imported from the web site http://www.global-view.com/forex-tradingtools/forex-history/ showing daily currency exchange rates for converting USD to JPY and GBP to JPY selected through the period from March, 13th, 2023 until, May 3rd 2024. The logarithms of the exchange rate ratio for two consecutive days are calculated, obtaining 300 values. For the given data set, using the formulas above, the following results are obtained:

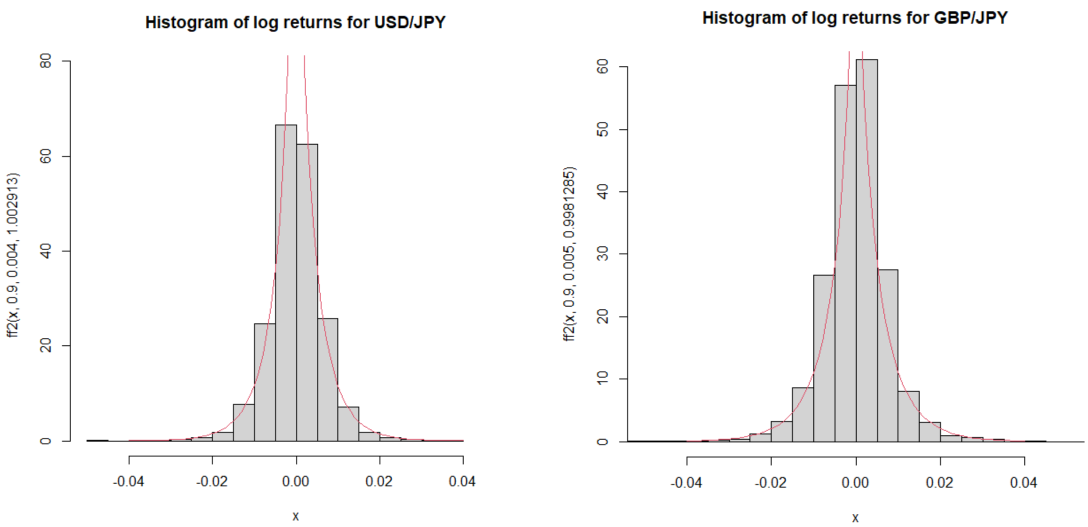
$$\alpha = 0.9\,, \quad \hat{\mathrm{m}} = 10^{-5}[-1.670 \quad 1.324] \quad \text{and} \quad \hat{\Sigma} = 10^{-5}\begin{bmatrix} 3.300 & 2.721 \\ 2.721 & 5.000 \end{bmatrix}.$$

Inserting the above values into $\kappa = \frac{\sqrt{m^2+4\tau^2}-m}{2\tau}$, and taking into account formulas in (8), the results for parameters κ and σ are calculated:

$$\kappa_{usd} = 1.002913\,, \ \kappa_{gbp} = 0.99812\,, \ \sigma_{usd} = 0.00406, \ \sigma_{gbp} = 0.005.$$

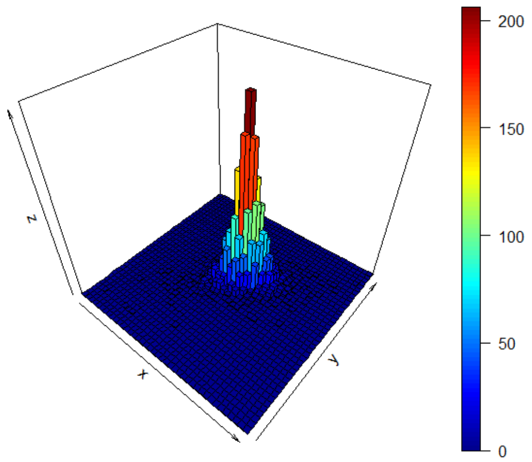
Obtained values of the parameters are substituted in the formula for density (4) and the following histograms overlayed by densities are presented proving that Weibull model given by (4) fits the data in a suitable way.

Figure 2
Histograms of log-returns transforming USD to JPY and GBP to JPY overlayed by proper densities



Source: Author's illustration

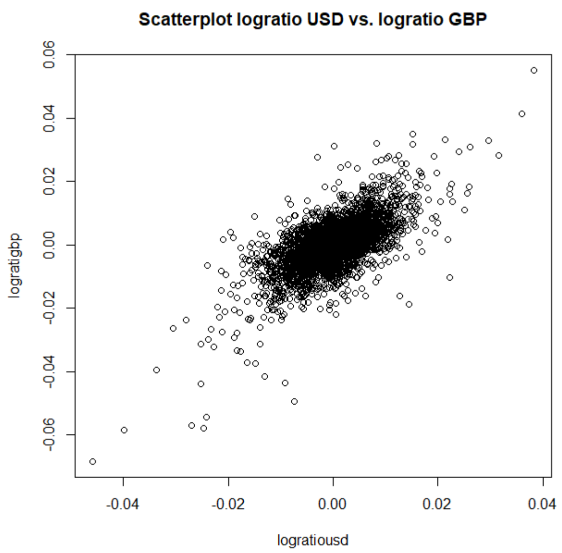
Figure 3
Three-dimensional histogram including 40 log- returns of the USD/JPY and GBP/JPY



Source: Author's illustration

The asymmetric pattern of the data can be clearly observed from the following chart indicating the modelling potential of the bivariate asymmetric Weibull distribution.

Figure 4
Scatterplot showing the asymmetric pattern of log-returns USD/JPY vs. log-returns GBP/JPY

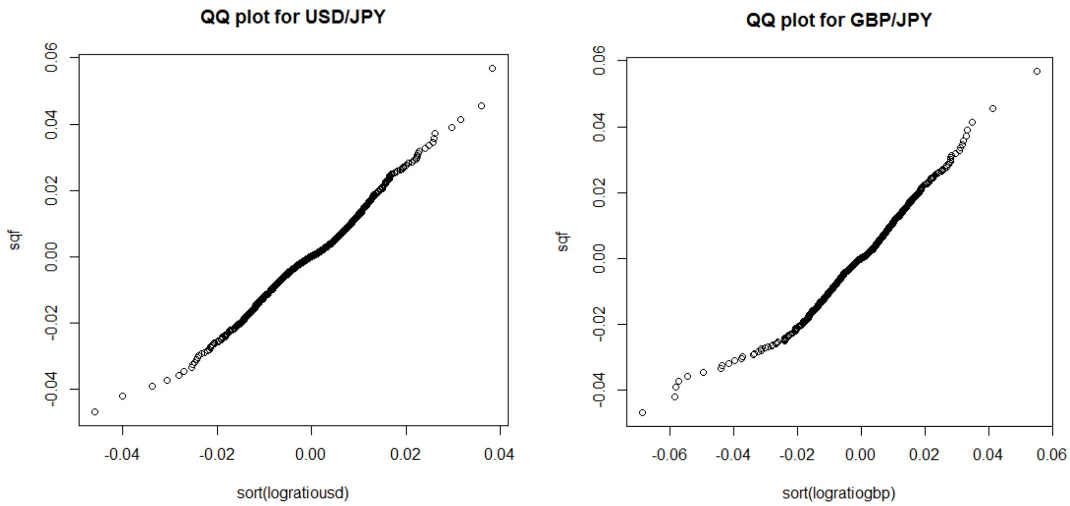


Source: Author's illustration

Discussion

To examine the fit of the distributions the QQ plots showing log returns of USD/JPY and GBP / JPY are presented. Overall, the graphs for individual directions show a good match but it can be clearly seen that the fit of QQ plot comparing log returns of USD/JPY is slightly better than the fit of log returns presented in GBP/JPY plot. The explanation may be the fact that the USD data set exhibits less variability than the GBP data set.

Figure 5
QQ plots comparing log-returns of USD/JPY and GBP/JPY



Source: Author's illustration

Comparison with previous research showed that a selection of the data does not change significantly the values of the parameters as well as the "nature" of graphical presentations of the data. The data sets exhibit the similar patterns.

Several empirical studies have shown that distributions of financial data are typically asymmetrical with heavy tails, even though, in this case, the bivariate data set does not imply a significant asymmetry which can be seen by observing that the estimated parameters $\boldsymbol{m}$ are close to zero.

Still, bivariate Weibull distribution proved to be a more suitable and reliable model for exchange rates, which suggests the usefulness of the new family of distributions in other circumstances as well.

Conclusion

Some basic facts about the asymmetric bivariate Weibull distribution are summarized along with the calculations of the parameters. Method of moments is used to estimate parameters. Most of estimators are given in closed form, while the estimation of the shape parameter $\boldsymbol{\alpha}$ required numerical search. All calculations are performed using statistical package R.

In order to derive proper results, a firm theoretical work was previously established. Estimation of the parameters required lengthy calculations forming a complicating formula that was coded in statistical package R. This procedure, including additional data package *nleqslv* (non- linear equation solver), enabled the calculation of the shape parameter α, as well as the calculation of other estimators that are subsequently found.

Beside the application in finance, there are still some other areas of studying that can be considered using the asymmetric bivariate Weibull distribution as an extension of the univariate, one – dimensional case. The main purpose of this extension to two-dimensional setting is a possibility to model joint distributions with known marginals. These models can be successfully used in other areas such as business, insurance and actuarial science,

Overall, bivariate distributions proved to be good models in financial modelling, but additional theoretical work is needed to apply these distributions in other areas of the real life.

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