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THE LONELY RUNNER PROBLEM FOR MANY RUNNERS

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Abstract. The lonely runner conjecture asserts that for any positive integer n and any positive numbers $v_1 < \cdots < v_n$ there exists a positive number t such that $||v_i t|| \geqslant 1/(n+1)$ for every $i = 1, \ldots, n$. We verify this conjecture for $n \geqslant 16342$ under assumption that the speeds of the runners satisfy $\frac{v_{j+1}}{v_j} \geqslant 1 + \frac{33 \log n}{n}$ for $j = 1, \ldots, n-1$.

1. Introduction

Let n be a positive integer, and let $v_1 < v_2 < \cdots < v_n$ be n positive real numbers. The *Lonely Runner Conjecture* asserts that there is a positive number t such that

$$||v_i t|| \geqslant \frac{1}{n+1} \tag{1.1}$$

for every $i = 1, 2, \ldots, n$. Throughout $||y||$ stands for the distance between a real number y and the nearest integer to y. Note that inequality (1.1) is optimal if, for instance, $v_i = vi$ for each $i = 1, \ldots, n$, where $v > 0$ is a fixed real number (see, e.g., [6]). For some n there are also other values of v_i's when equality in (1.1) is attained (see [12]).

The conjecture originally comes from the paper of Wills ([17]), where it is stated for integer v_i's. Independently, this problem was considered by Cusick ([7]). The name of the lonely runner conjecture comes from the following beautiful interpretation of the problem due to Goddyn ([5]). *Suppose k runners having distinct constant speeds start at a common point and run laps on a circular track with circumference 1. Then for any given runner there is a time at which that runner is at least $1/k$ (along the track) away from every*

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other runner. Taking $k = n+1$ and assuming that the speeds of the runners are $u_0 < u_1 < \cdots < u_n$, we see that at the time $t > 0$ the runner with speed, say, u_0 is at distance $\geqslant 1/(n+1)$ from all other runners if and only if (1.1) holds for $v_i = u_i - u_0$, $i = 1, \dots, n$.

It seems that the lonely runner conjecture is very deep in general. It is known that it has some useful applications to so-called view-obstruction problems, flows in regular matroids, chromatic numbers for distance graphs, etc. ([5, 7, 8, 13]). The problem has been settled for $n = 2, 3$ ([4]), $n = 4$ (first in [8] and then a simpler proof was found in [5]), $n = 5$ ([6], a simpler proof in [15]). Recently, Barajas and Serra ([2]) proved the conjecture for $n = 6$. For each $n \geqslant 7$ the lonely runner conjecture is still open.

On the other hand, there are some conditions on the speeds of the runners $v_1 < \cdots < v_n$ under which the lonely runner conjecture holds. If, for example,

$$v_n/v_1 \leqslant n \tag{1.2}$$

then taking $t_0 = 1/(n+1)v_1$ it is easy to see that the numbers $v_i t_0 = v_i/(n+1)v_1$, $i = 1, \dots, n$, all lie in the interval $[1/(n+1), n/(n+1)]$, so (1.1) holds.

Recently, Pandey ([14]) showed that the condition

$$\frac{v_{j+1}}{v_j} \geqslant \frac{2(n+1)}{n-1} \tag{1.3}$$

for each $j = 1, \dots, n-1$ implies (1.1). This inequality (in a slightly different form) was also obtained by Ruzsa, Tuza and Voigt ([16]), and then the constant $2(n+1)/(n-1)$ was improved to 2 in [3]. Using the same method of nested intervals as in [14] one can easily prove (1.1) under condition

$$\frac{v_{j+1}}{v_j} \geqslant \frac{2n}{n-1} \tag{1.4}$$

for $j = 1, \dots, n-1$ which is slightly weaker than (1.3) (see the beginning of Section 2).

In this note we prove the following:

THEOREM 1.1. *Let $n \geqslant 32$, and let $v_1 < v_2 < \cdots < v_n$ be positive real numbers satisfying*

$$v_{j+[(n+1)/12e]} \geqslant (n+1)v_j$$

for each $j = 1, 2, \dots, n - [(n+1)/12e]$. Then there is a positive number t such that $||v_i t|| > 1/(n+1)$ for each $i = 1, 2, \dots, n$.

Here and below, $[y]$ stands for the integral part of a real number y. Theorem 1.1 implies the following improvement of the conditions (1.3), (1.4) under which (1.1) holds:

COROLLARY 1.2. *Suppose that κ is a constant strictly greater than $8e = 21.74625\dots$. Then there is a positive integer $n(\kappa)$ such that for each integer*

$n \geqslant n(\kappa)$ and each collection of n positive numbers $v_1 < v_2 < \cdots < v_n$ satisfying

$$\frac{v_{j+1}}{v_j} \geqslant 1 + \frac{\kappa \log n}{n} \tag{1.5}$$

for every $j = 1, 2, \ldots, n-1$ there is a positive number t such that

$$||v_i t|| > 1/(n+1) \tag{1.6}$$

for every $i = 1, 2, \ldots, n$. In particular, for $\kappa = 33$, one can take $n(33) =$ 16342.

Note that the condition

$$\frac{v_{j+1}}{v_j} \geqslant 1 + \frac{22 \log n}{n},$$

$j = 1, \ldots, n-1$, of Corollary 1.2 with $\kappa = 22 > 8e$ yields $v_n/v_1 \geqslant (1 + \frac{22 \log n}{n})^{n-1}$. Here, the right hand side is approximately n^{22} for large n. Comparing with (1.2) we see that there is still a polynomial gap between n and n^{22} for the bounds on v_n/v_1 for which the lonely runner conjecture is not verified. At least this gap is smaller than a corresponding exponential gap between n and (roughly) 2^{n-1} which comes from (1.3) and (1.4).

We remark that, by Lemma 6 in [10], for any positive numbers $v_1 < \cdots < v_n$ and any $\varepsilon > 0$ and $T > 0$ there is an interval $I = [u_0, u_0 + \varepsilon/2v_n]$, where $u_0 > T$, such that

$$||v_i t|| < \varepsilon$$

for each $t \in I$ and each $i = 1, \ldots, n$. This shows that all the runners can be arbitrarily close to their starting position at arbitrarily large time t. The referee pointed out that this problem is somewhat related to Bogolyubov's theorem on Bohr neighborhoods and, despite some similarity to the lonely runner, has a different nature.

We shall derive Theorem 1.1 from Lemma 2.1 below. Since the proof of the lemma is based on a so-called Lovász local lemma (see [1] and [11]), the Lovász lemma is implicitly present in the proofs below.

2. Proofs

We first prove that (1.4) implies (1.1). Indeed, setting $I_1 := [1/(n+1)v_1, n/(n+1)v_1]$ we see that $||v_1 t|| \geqslant 1/(n+1)$ for each $t \in I_1$. Put $k_1 := 0$. We claim that there is a sequence of nested intervals $I_1 \supseteq \cdots \supseteq I_n$ of the form $I_i := [(k_i + 1/(n+1))/v_i, (k_i + n/(n+1))/v_i]$ with integer k_i for $i = 1, \ldots, n$. Then $||v_i t|| \geqslant 1/(n+1)$ for each $i = 1, \ldots, n$. The proof is by induction. Assume that we have such sequence of nested intervals $I_1 \supseteq \cdots \supseteq I_j$, where $1 \leqslant j \leqslant n-1$. Note that the interval

$$\Big[\frac{v_{j+1}k_j}{v_j} + \frac{v_{j+1}/v_j - 1}{n+1}, \frac{v_{j+1}k_j}{v_j} + \frac{n(v_{j+1}/v_j - 1)}{n+1}\Big]$$

contains a positive integer, say, k_{j+1}, because the length of this interval is $\geqslant 1$, by (1.4). From

$$\frac{v_{j+1}k_j}{v_j}+\frac{v_{j+1}/v_j-1}{n+1}\leqslant k_{j+1}\leqslant\frac{v_{j+1}k_j}{v_j}+\frac{n(v_{j+1}/v_j-1)}{n+1}$$

we deduce that $I_{j+1}:=[(k_{j+1}+1/(n+1))/v_{j+1},(k_{j+1}+n/(n+1))/v_{j+1}]\subseteq I_j$. This completes the induction step and so proves that (1.4) implies (1.1).

The next lemma is Theorem 1.1 with dimension $m=1$ from [9].

LEMMA 2.1. *Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of real numbers. If h is a positive integer, $c(h)$ is a real number greater than $4eh$ and $(t_k)_{k=1}^{\infty}$ is a sequence of positive numbers satisfying $t_{k+h}\geqslant c(h)t_k$ for each integer $k\geqslant 1$ then there is a real number x such that*

$$||t_kx-\xi_k||>\frac{1}{8eh}-\frac{1}{2c(h)}$$

for every $k\geqslant 1$.

Take $\xi_k:=0$ for each $k\geqslant 1$. Put $t_k:=v_k$ for $k=1,\dots,n$ and, say, $t_k:=v_nc(h)^{k-n}$ for $k\geqslant n+1$. By Lemma 2.1, there is a real number x such that

$$||v_kx||>\frac{1}{8eh}-\frac{1}{2c(h)}\tag{2.1}$$

for $k=1,\dots,n$. Since $||y||=||-y||$, the same inequality holds for $t:=|x|>0$ instead of x. Obviously, $x\neq 0$, so $t>0$.

Put $h:=[(n+1)/12e]$ and $c(h):=n+1$. Note that $h\geqslant 1$, because $n\geqslant 32$. Since $e\notin\mathbb{Q}$, we have $h<(n+1)/12e$. Thus the right hand side of (2.1) is

$$\frac{1}{8eh}-\frac{1}{2c(h)}=\frac{1}{8e[(n+1)/12e]}-\frac{1}{2(n+1)}>\frac{12e}{8e(n+1)}-\frac{1}{2(n+1)}=\frac{1}{n+1}.$$

Therefore, the inequality $||v_it||>1/(n+1)$ holds for $i=1,\dots,n$ provided that $v_{i+h}\geqslant(n+1)v_i$ for $i=1,\dots,n-h$. This is exactly the condition of Theorem 1.1. The proof of the theorem is completed.

We first prove that one can take $n(33)=16342$ in Corollary 1.2. Assume that inequality (1.5) holds with $\kappa=33$. To apply Theorem 1.1 we will check with Maple that

$$\Big(1+\frac{33\log n}{n}\Big)^h=\Big(1+\frac{33\log n}{n}\Big)^{[(n+1)/12e]}>n+1$$

for each integer $n\geqslant 16342$. Indeed, the function

$$g(z):=\Big[\frac{z+1}{12e}\Big]\log\Big(1+\frac{33\log z}{z}\Big)-\log(z+1)$$

is positive for $z\geqslant 16342$ except for two intervals J_1 and J_2 such that $J_1\subset(16373,16374)$ and $J_2\subset(16406,16407)$. At the points $z=$

$16373, 16374, 16406, 16407$ the function $g(z)$ is positive. Thus $g(n) > 0$ for each integer $n \geqslant 16342$.

For the proof of Corollary 1.2 we assume that (1.5) holds with some $\kappa > 8e$. We shall derive inequality (1.6) directly from Lemma 2.1. Set $\epsilon := (k - 8e)/(4e + 1)$. Then $\epsilon > 0$ satisfies

$$8e(1 + \epsilon/2) = \kappa - \epsilon. \tag{2.2}$$

This time, we select $h := [(n + 1)/(\kappa - \epsilon)]$ and $c(h) := [(n + 1)/\epsilon] + 1$. Then, by (2.2), the right hand side of (2.1) is

$$\begin{aligned} \frac{1}{8eh} - \frac{1}{2c(h)} &= \frac{1}{8e[(n+1)/(\kappa - \epsilon)]} - \frac{1}{2([(n+1)/\epsilon] + 1)} \\ &> \frac{\kappa - \epsilon}{8e(n+1)} - \frac{\epsilon}{2(n+1)} = \frac{1}{n+1}. \end{aligned}$$

Hence, by Lemma 2.1, inequality (1.6) holds for every $i = 1, \ldots, n$ and some $t > 0$ provided that $v_{i+h} \geqslant ([(n + 1)/\epsilon] + 1)v_i$ for each $i = 1, \ldots, n - h$. Note that (1.5) implies $v_{i+h} \geqslant (1 + \frac{\kappa \log n}{n})^h v_i$ for $i = 1, \ldots, n - h$. Since $h \geqslant 1$ for $n \geqslant \kappa$, it remains to prove the inequality

$$\Big(1 + \frac{\kappa \log n}{n}\Big)^{[(n+1)/(\kappa - \epsilon)]} \geqslant [(n + 1)/\epsilon] + 1 \tag{2.3}$$

for each sufficiently large n.

It is clear that $\kappa/(\kappa - \epsilon) > 1 + \epsilon/\kappa$. Thus there is a positive integer $n_1 = n_1(\epsilon, \kappa) = n_1(\kappa)$ such that the left hand side of (2.3) is greater than $n^{1+\epsilon/\kappa}$ for $n \geqslant n_1$. On the other hand, there is a positive integer $n_2 = n_2(\epsilon, \kappa) = n_2(\kappa)$ such that the right hand side of (2.3) is at most $2n/\varepsilon < n^{1+\epsilon/\kappa}$ for $n \geqslant n_2$. Thus (2.3) holds for each $n \geqslant \max(n_1(\kappa), n_2(\kappa))$. This completes the proof of the corollary.

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